Lecture 9

- no problem set this week. Next problem session
- proof of Tychonoff's theorem.

Recall :- X is connected if every continuous
$f : X \longrightarrow \{0,1\}$ is constant.

$$\Updownarrow$$

The only sets which are both open and closed
are X and ϕ.

→ connectedness is a topological property.

→ X Path-connected if $\forall\ x, y \in X$, $\exists$ continuous
$\gamma : [0,1] \longrightarrow X$ w/ $\gamma(0) = x$ and $\gamma(1) = y$.
γ is called a path b/w x and y.

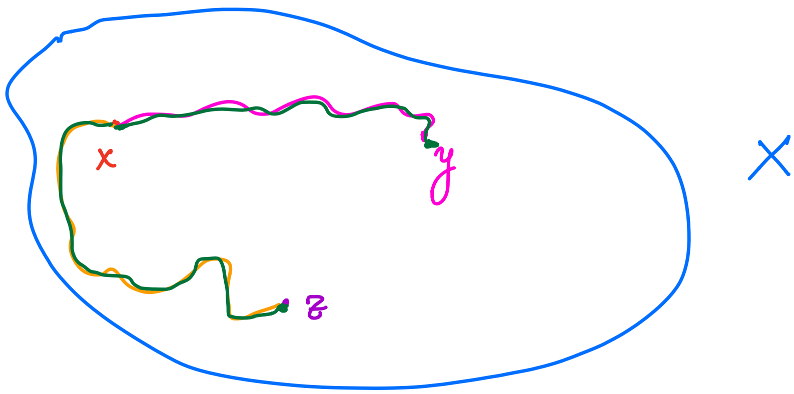
→ Path connected $\Longrightarrow$ connected. Converse NOT true

Theorem Path-connectedness is a topological property.
(continuous image of a path-connected space is path-connected).

Proof Exercise.

Theorem:- X is path-connected $\Longleftrightarrow$ $\exists\, x \in X$ s.t. any other point in X can be joined to x.

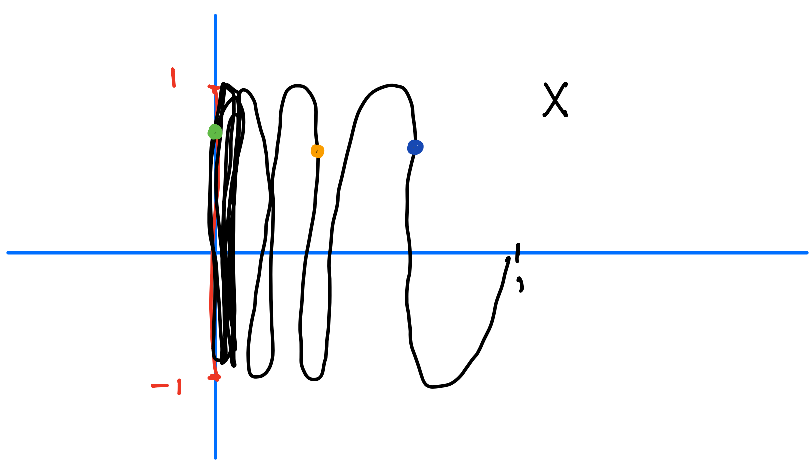
Proof



example Topologist's sine curve

$X = A \cup B \subset \mathbb{R}^2$

$A = \{ (x, \sin(\pi/x)) \mid 0 < x \le 1 \}$ $\quad B = \{ (0,y) \mid -1 \le y \le 1 \}$



X is connected but not path-connected.

A is actually path connected $\Rightarrow$ connected.

$\bar{A} = X$. (Check)

(Last week :- $A \subset B \subset \bar{A}$ then A is connected $\Rightarrow$ B is connected).

$\bar{A} = X$ is connected.

X is not path-connected. In particular, there is no path b/w $(0,0)$ and $(1,0)$.

Suppose $\exists$ $\gamma : [0,1] \longrightarrow X$ joining $(0,0)$ & $(1,0)$.

$\gamma(t) = (\gamma_1(t), \gamma_2(t))$. B is closed in X

$\Rightarrow$ $\gamma^{-1}(B)$ closed in $[0,1]$ and $0 \in \gamma^{-1}(B)$

as $\gamma(0) = (0,0) \in B$.

Let t_0 be the least upper bound of the closed and bounded set $\gamma^{-1}(B)$. $\Rightarrow$ $t_0 \in \gamma^{-1}(B)$

$\Rightarrow$ $\gamma(t_0) \in B$. $0 < t_0 < 1$.

$\Rightarrow$ $\gamma_2(t_0) \in [-1,1]$, WLOG, let $\gamma_2(t_0) \leq 0$.

Claim :- γ_2 is not continuous at t_0.

For any $\delta > 0$ w/ $t_0 + \delta \leq 1$ we must have

$\gamma_1(t_0 + \delta) > 0$. $\Rightarrow$ $\exists$ $n \in \mathbb{N}$ s.t.

$$\gamma_1(t_0) < \frac{2}{4n+1} < \gamma_1(t_0+\delta).$$

$\because$ γ_1 is continuous $\Rightarrow$ By the intermediate value theorem $\exists$ t w/ $t_0 < t < t_0+\delta$

s.t. $\gamma_1(t) = \dfrac{2}{4n+1}$. $\Rightarrow$ $\gamma_2(t) = \sin\left(\dfrac{\pi}{2}(4n+1)\right)$

$= 1$

$\Rightarrow$ $|\gamma_2(t) - \gamma_2(t_0)| \geq 1$ which is not possible

as $|t-t_0| < \delta$.

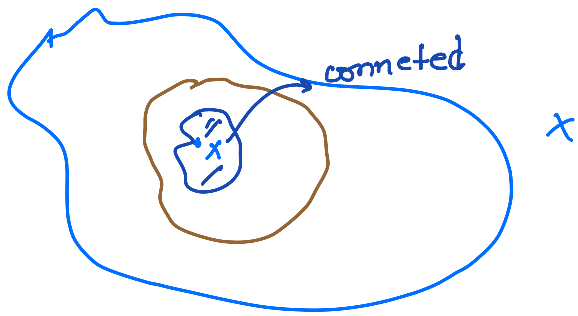
$\Rightarrow$ γ_2 is not continuous. $\Rightarrow$ X is not path-connected.

Remark:- • If $\gamma_2(t_0) \geq 0$ then choose $n \in \mathbb{N}$ s.t

$$\gamma_1(t_0) \leq \frac{2}{4n-1} \leq \gamma_1(t_0+\delta).$$

• The main point was that if $\gamma(0) = (0,0)$ Then $\gamma_1(t) = 0$ $\forall$ $t \in [0,1]$. But that won't be the case if we assume the existence of a path b/w $(0,0)$ and $(1,0)$.

Def^n A space X is locally connected if $\forall$ $x \in X$, every nbd of x contains a connected nbd of x.



Def^n A space X is locally path-connected if $\forall\, x \in X$ every nbd of x contains a path-connected nbd.

Union of disjoint balls $\rightsquigarrow$ locally connected space which is NOT connected.

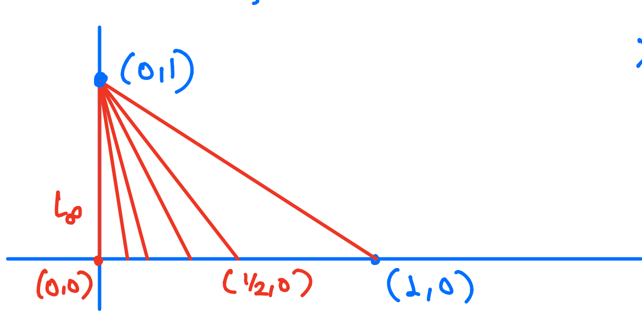
Local path-connectedness $\Rightarrow$ local connectedness.

* The topologist's sine curve is not locally connected but it is connected.

$B \subset X$.

$(0,y) \in X \ , \ -1 < y < 1$

* Space which is path-connected but not locally path-connected.



$$X = \left(\bigcup_{n=1}^{\infty} L_n\right) \cup L_\infty$$

L_n = straight line segment from

$(0,1)$ to $(1/n, 0)$

X is path-connected.

small neighbourhoods of $(0,0)$ is never going to be connected

Theorem :- If X is connected and locally path-connected then X is also path-connected.

Proof :- Series of exercises in PSET4.

Algebraic Topology

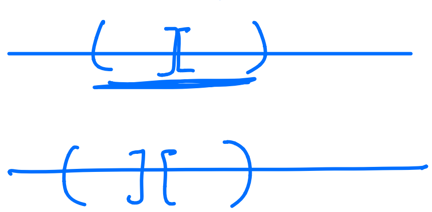
When are two given spaces homeomorphic?

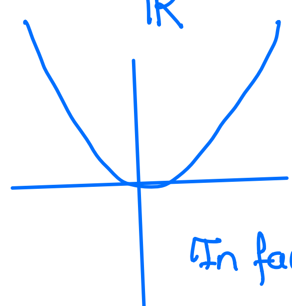
$[0,1]$ $\not\cong$ $(0,1)$

compact   non compact

$\mathbb{R}$ $\not\cong$ $\mathbb{R}^2$

as $\mathbb{R} \setminus \{0\}$ is not path-connected



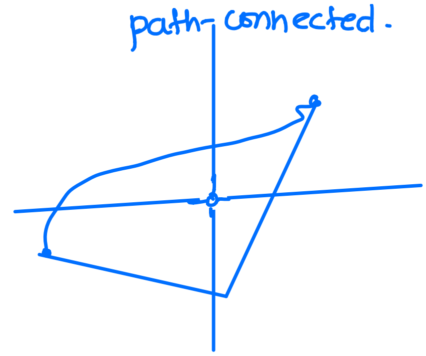


$\mathbb{R}^2 \setminus \{(0,0)\}$ is path-connected.

In fact, $\mathbb{R}^2 \setminus C$ where C is countable is

$\mathbb{R}$

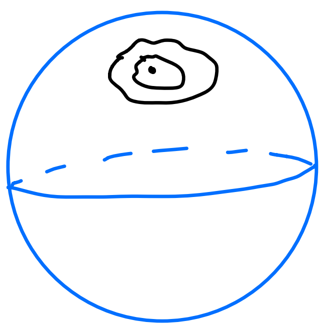
path-connected.



$\mathbb{R}^2 \not\cong \mathbb{R}^3$

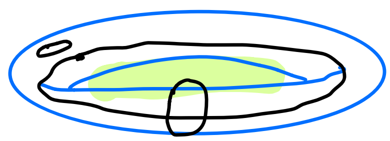
$\mathbb{R}^n \not\cong \mathbb{R}^m$

$n \neq m$.

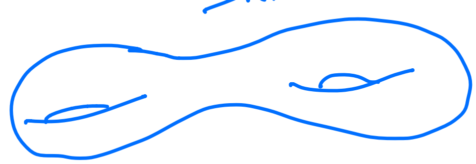


$S^2 \not\cong$

$S^1 \times S^1$



$\not\cong$



Fundamental group of a topological space.

Group.

$\pi_1(X)$ = fundamental group of X