Lecture 8

Connectedness & Path-connectedness

$\mathbb{R} \rightsquigarrow$ "connected"

$(-\infty, 1) \cup (2, \infty) \rightsquigarrow$ "disconnected"

Def 1 A space X is connected if every continuous map $X \longrightarrow \{\pm 1\}$ is constant.

Def 2 A space X is connected if and only if the only open and closed subsets of X are X and ϕ.

$$\Updownarrow$$

If X is connected then $\nexists$ two disjoint open sets A, B $(A \cap B = \phi)$ w/ $A, B \neq \phi$ s.t. $X = A \cup B$.

- A subset $A \subset X$ is connected if it is connected in the subspace topology.

Thm Def(1) $\Longleftrightarrow$ Def 2.

Proof

Suppose X is connected w.r.t. Def 1. Suppose Def(2) doesn't hold $\Rightarrow$ $\exists$ $X_0 \subset X$ which is both open and closed in X, $X_0 \neq X, \Phi$.

define $f : X \longrightarrow \{\pm 1\}$ by $f|_{X_0} = -1$

and $f|_{X \setminus X_0} = +1$. f is not constant and is continuous.

$f^{-1}(\{\pm 1\}) = X$
$f^{-1}(\phi) = \phi$
$f^{-1}(\{-1\}) = X_0$ open
$f^{-1}(\{+1\} = X \setminus X_0$ is open as X_0 is closed.

$\Rightarrow$ Def (1) $\Longrightarrow$ (Def 2).

Assume def 2. and let
$f : X \longrightarrow \{\pm 1\}$ be continuous.
If f were not constant then
$f^{-1}(-1)$ and $f^{-1}(1)$ would be non-empty open and closed subsets of X which would be disjoint.
$\Longrightarrow$ f must be constant.

$\square$

Rem:- X is connected iff every cont. map $X \to \{0,1\}$ is constant. We're using the fact that we have discrete topology on $\{0,1\}$.

Ex. 1) $\mathbb{R}$ is connected., open in Y open in Y in subspace top.

2) $Y = [-1, 0) \cup [0, 1] \subset \mathbb{R}$ is disconnected.

$= (-2, 0) \cap Y$ $\quad$ $(0, 2) \cap Y$

3) I is an interval in $\mathbb{R}$. I is connected.

Proof $f : I \to \{\pm 1\}$ continuous & non-constant

then let $f(x) = 1$, $f(y) = -1$, $x, y \in I$. Let $x < c < y$

$f(c) = 0$ by the Intermediate value thm but that is not possible.

$\Rightarrow$ f must be constant $\Rightarrow$ any interval I of $\mathbb{R}$ is connected.

4) $\mathbb{Q} \subset \mathbb{R}$ is not connected.

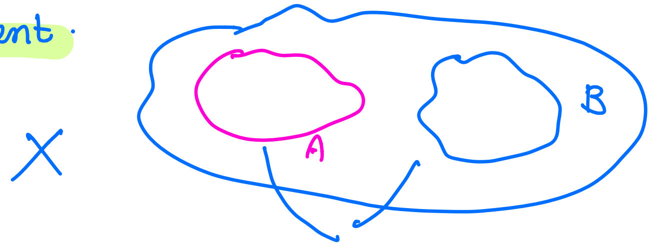
$p \in \mathbb{Q}$, $q \in \mathbb{Q}$ $\exists$ $i \in \mathbb{R} \setminus \mathbb{Q}$

$p < i < q$

$$\mathbb{Q} = (-\infty, i) \cup (i, \infty)$$

The only connected subsets of $\mathbb{Q}$ are singletons.

Any maximal connected subset of a space X is called a connected component.



connected.

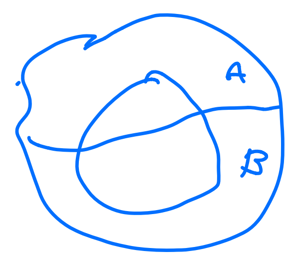
Prop: Any two connected components $A, B \subset X$ are either identical or disjoint.

Proof If $A = B$ then we're done. If not, let's assume that $A \cap B \neq \phi$. We claim :- $A \cup B$ is connected.

$f : A \cup B \to \{\pm 1\}$ is continuous.

$\Rightarrow$ $f_{|A}$ must be constant and $\exists\, x \in A \cap B$

$f_{|B}$ must be constant



$\Rightarrow$ $f_{|A \cup B}$ is also constant.

$\Rightarrow$ $A \cup B$ is connected.

a contradiction to the maximality of A and B. $\Rightarrow$ $A \cap B = \phi$.

$\square$

Theorem :-

(i) X is a topological space. Let A and B be connected subsets of X s.t. $A \cap B \neq \phi$. Then $A \cup B$ is connected.

ii) A be a connected subset of X. Let $A \subset B \subset \bar{A}$.

Then B is connected.

iii) $\{A_i\}_{i \in I}$ be a collection of connected subsets of X w/ the property that $\forall\, i, j \in I$, $A_i \cap A_j \neq \phi$. Then $A = \bigcup_{i \in I} A_i$ is connected.

<u>Proof</u> :- ii) Suppose $A \subset B \subset \bar{A}$, let $f : B \to \{\pm 1\}$ be continuous. $f_{|A}$ must be constant as A is connected, so let $f_{|A} = 1$. Let $b \in B$, $\{f(b)\}$ is an open set in $\{\pm 1\}$ $\exists$ an open set $U \ni b$ s.t $f(U) \subset \{f(b)\}$.

$\because$ $b \in B \subset \bar{A}$ $\Rightarrow$ b is a cluster point / limit point of A. $\Rightarrow$ $\exists$ $a \in A \cap U$.

$\Rightarrow$ $f(a) \in \{f(b)\}$ or $f(b) = f(a) = 1$.

$\Rightarrow$ f is constant on B.

$\Rightarrow$ B is connected. $\boxtimes$

<u>Corr:</u> Closure of a connected set is connected set.

Theorem:- Connectedness is a topological property, i.e., if $f: A \to B$ is continuous w/ A connected $\Rightarrow$ $f(A)$ is connected in B.

Proof. $g: f(A) \longrightarrow \{\pm 1\}$ continuous.

$h: A \longrightarrow \{\pm 1\}$ continuous must be constant

$g \circ h: A \longrightarrow \{\pm 1\}$ must be constant on A

$\Rightarrow$ g must be constant on $f(A)$

$\Rightarrow$ $f(A)$ is connected. $\boxtimes$

Remark:- In order to show that a set is disconnect-ed, it sufficises to produce a continuous function from that set to $\{\pm 1\}$ which is non-constant.

Examples:-

1) $GL(2,\mathbb{R}) = \left\{ \begin{bmatrix} a, & b \\ c & d \end{bmatrix} \middle| \, ad - bc \neq 0 \right\}$

is <u>not</u> connected.

$f: GL(2,\mathbb{R}) \longrightarrow \{\pm 1\}$ not constant, continuous.

$f = \det: GL(2,\mathbb{R}) \longrightarrow \{\pm 1\}$ is a continuous function as it is the composition of continuous operations of multiplication & substraction.

det. is not a constant function on $GL(2,\mathbb{R})$

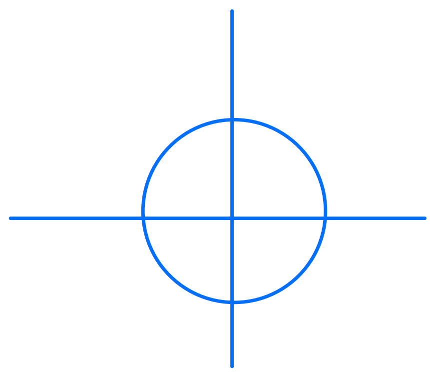
$\Rightarrow$ $GL(2,\mathbb{R})$ is disconnected.

2) $O(n,\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det(A) = \pm 1 \}$ is disconnected.

3) S^1, circle in $\mathbb{R}^2$

$\shortparallel$

$\{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2=1 \}$

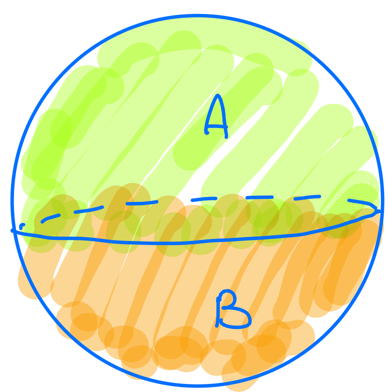


$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x^2+y^2-1$

continuous image of a connected set is connected.

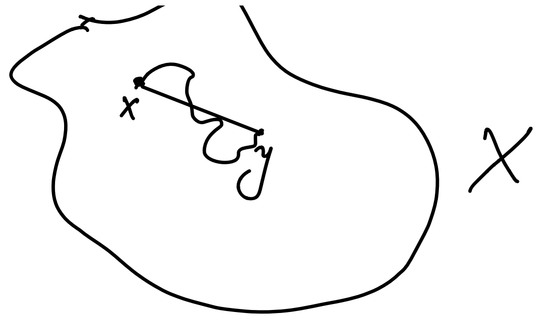
ii) S^n is connected.



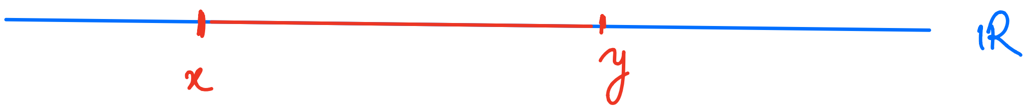
Hemispheres A, B are connected, $A \cap B \neq \phi$ and $S^n = A \cup B$.

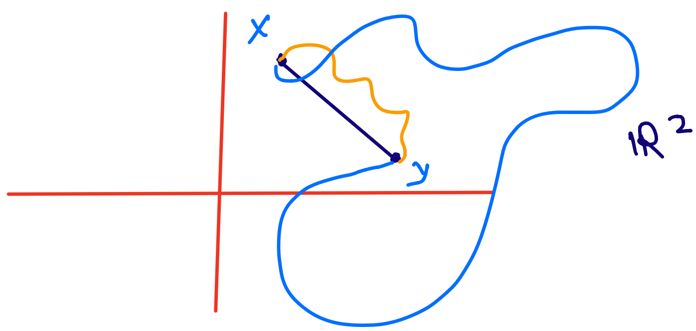
$\Rightarrow$ S^n is connected.

Path-Connectedness



Def^n A space X is **path-connected** if for every pair of points $x, y \in X$ $\exists$ a continuous map $\gamma : [0,1] \longrightarrow X$ s.t
$\gamma(0) = x$ and $\gamma(1) = y$.
Such a γ is called a path from x to y.





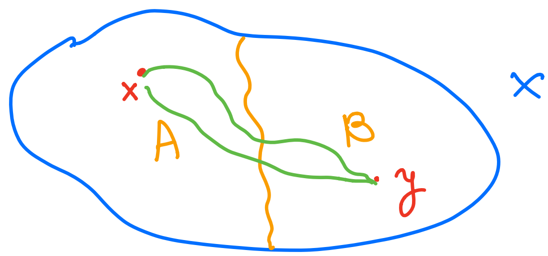
Any maximal path-connected subset of X is called a **path-component**.

A subset $A \subset X$ is path-connected if it is so in the subspace topology, i.e. $\forall$ $a, b \in A$ $\exists$ exist a path joining a and b which lies completely in A.

Theorem:- Every path-connected space is connected.

Remark:- The converse is not true.

Proof:-



Suppose X is path-connected and not connected.

$\Rightarrow \exists \; x, y \in X$ and a continuous function

$$f : X \to \{0,1\} \text{ s.t. } f(x) = 0, \; f(y) = 1.$$

$\because$ X is path-connected $\Rightarrow \exists$ continuous map

$$\gamma : [0,1] \longrightarrow X \text{ s.t } \gamma(0) = x, \; \gamma(1) = y.$$

$$g = f \circ \gamma : [0,1] \to \{0,1\} \text{ s.t } g(0) = 0 \text{ and } g(1) = 1$$

which can't be true as $[0,1]$ is connected.

$\boxtimes$

The converse of above theorem is NOT true.

Topologist's sine curve is a counter-example

$$A = \{ (x, \sin(\pi/x)) \mid 0 < x \leq 1 \} \cup B = \{(0,y) \mid -1 \leq y \leq 1\}$$

$\subseteq \mathbb{R}^2$