Lecture 7

- Problem Set 3 will be uploaded after today's Prob. session.

Recall:- $p: X \to Y$ quotient map if $U \subset Y$ is open

$\Longleftrightarrow$ $p^{-1}(U) \subset X$ is open.

$p: X \to A$ is a surjective map, the strongest topology on A w.r.t. which p is a quotient map is the quotient topology on A, induced by p.

U is open in A if $p^{-1}(U)$ is open in X.

→ $p: X \to X^*$ ~ partitions of X, quotient topology induced by p on X^* makes it a quotient space of X.

Theorem Let $p: X \to Y$ be a quotient map. Let Z be another space and let $g: X \to Z$, g is const-

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

-ant on each set $p^{-1}(\{y\})$, $y \in Y$.

Then g induces a map $f : Y \to Z$ s.t $f \circ p = g$.
f is continuous $\Leftrightarrow$ g is continuous.
f is a quotient map $\Leftrightarrow$ g is a quotient map.

<u>Proof</u> If $y \in Y$, $g(p^{-1}(\{y\}))$ is a one-point set in Z, say $\{z\}$.
Let $f(y) = z$. So we have $f : Y \to Z$.
s.t. $\forall$ $x \in X$, $f(p(x)) = g(x)$; i.e. $f \circ p = g$.

Suppose f is continuous $\Rightarrow$ $f \circ p = g$ is also continuous.
Conversely, let g be continuous. Let $\underset{\text{open}}{V \subset Z}$

$\Rightarrow$ $g^{-1}(V)$ is open in X.

$g^{-1}(V) = p^{-1}(f^{-1}(V))$

$\because$ p is a quotient map. $\Rightarrow$ $f^{-1}(V)$ must be open in Y. $\Rightarrow$ f is continuous.

Similarly one proves the 2nd part.

$\boxtimes$

<u>Theorem</u> :- Let $g : X \to Z$ be a continuous surjective map. Let $X^* = \{ g^{-1}(\{z\}) \mid z \in Z \}$ and give it the quotient topology.

(a) g induces a bijective continuous map $f : X^* \to Z$ which is a homeomorphism $\iff$ g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, dashed, "f"] & Z \end{tikzcd}$$

(b) If Z is Hausdorff, so is X^*.

<u>Proof</u> :- $g \rightsquigarrow f : X^* \to Z$ is continuous.

f is bijective.

Let f is a homeomorphism. f and $p : X \to X^*$ are quotient maps $\Rightarrow$ $f \circ p = g$ is a quotient map.

Conversely, let g be a quotient map $\Rightarrow$ f is a quotient $\Rightarrow$ f is a homeomorphism.

(b) Z is Hausdorff.

Let x_1 and x_2 be distinct points in X^*

$f(x_1)$ and $f(x_2)$ are distinct in Z.

$U \ni f(x_1)$, $V \ni f(x_2)$, $U \cap V = \phi$.

$\Rightarrow$ $f^{-1}(U) \ni x_1$, $f^{-1}(V) \ni x_2$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$.

Compactness

Def^n $A \subset X$ is sequentially compact if every sequence in A has a subsequence that converges to a point A.

Lemma :- If $(x_n) \in X$ is a sequence w/ a cluster point $x \in X$ and x has a countable nbd base then (x_n) has a subsequence converging to x.

Corr :- If X is compact & first countable $\Rightarrow$ X is also sequentially compact.

Proof of the Lemma :-

WLOG, assume that the countable nbd base at x forms a nested sequence

$$X \supset U_1 \supset U_2 \supset U_3 \ldots \ni x$$

$\because$ x is a cluster point $\Rightarrow$ $\exists$ $k_1 \in \mathbb{N}$ s.t. $x_{k_1} \in U_1$

so $\exists$ $k_n \in \mathbb{N}$ s.t. $x_{k_n} \in U_n$, $k_n > k_{n-1}$.

$\Rightarrow$ (x_{k_n}) is a subsequence of (x_n)

$(x_{k_n}) \longrightarrow x$ $\Rightarrow$ (x_{k_n}) is the desired subsequence. $\square$

For proving sequential compactness $\Rightarrow$ compactness, we'll need the 2^{nd} countability axiom.

Lemma :- Let X be a 2^{nd} countable space. Then every open cover of X has a countable subcover.

Proof- Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for X.

$\mathcal{B}$ is a countable base. $\Rightarrow$

each $U_\alpha = \bigcup_{B \in \mathcal{B}'' \subset \mathcal{B}} B$

and the collection of sets in $\mathcal{B}$ that are contained in some U_α is a countable subcollection $\mathcal{B}' \subset \mathcal{B}$ also covers X.

$$\mathcal{B}' = \{V_1, V_2, V_3, \dots\}$$

We can choose $\forall$ $V_n \in \mathcal{B}'$ an element $\alpha_n \in I$ s.t $V_n \subset U_{\alpha_n}$ and $\{U_{\alpha_n}\}_{n \in \mathbb{N}}$ is a countable subcover of $\{U_\alpha\}_{\alpha \in I}$.

Theorem :- If X is a second countable and sequentially compact then it is compact.

Proof. Suppose $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X. We want to prove that it admits a finite subcover.

From the previous lemma, $\because$ X is 2^{nd} countable

$\Rightarrow$ $\{U_\alpha\}_{\alpha\in I}$ admits a countable subcover $\{U_i\}_{i\in\mathbb{N}}$.

Now we proceed by contradiction. Suppose $\nexists$ $n\in\mathbb{N}$ s.t. $\{U_1, U_2, \ldots, U_n\}$ covers X.

$\Rightarrow$ $\exists$ a sequence $x_n\in X$ s.t.

$$x_n \notin U_1\cup U_2\cup\ldots\cup U_n. \qquad \text{——— } (1)$$

$\because$ X is sequentially compact $\Rightarrow$ some subsequence $(x_{k_n}) \longrightarrow x\in X$.

$\because$ $\{U_i\}_{i\in\mathbb{N}}$ is again a cover of X $\Rightarrow$ $x\in U_N$ for some $N\in\mathbb{N}$.

$\Rightarrow$ x_{k_n} also lies in U_N $\forall$ n large enough.

This contradicts (1) when $k_n \geq N$.

$\Rightarrow$ $\{U_\alpha\}_{\alpha\in I}$ admits a finite subcover

$\Rightarrow$ X is compact.

$\blacksquare$

Tychonoff's Theorem

$\{X_\alpha\}_{\alpha \in I}$, X_α is compact $\forall \alpha \in I$.

What can we say about $\prod_{\alpha \in I} X_\alpha$?

Theorem (Tychonoff) For any collection $\{X_\alpha\}_{\alpha \in I}$ of compact spaces, $\prod_{\alpha \in I} X_\alpha$ is compact in product topology.

$\Updownarrow$

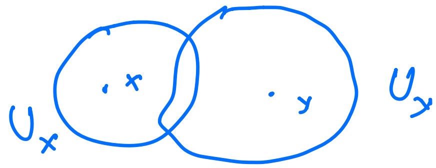
Zorn's Lemma

Separation Axioms

Def^n X is said to satisfy axiom T_0 if for every pair of distinct points $x, y \in X$ $\exists$ an open set of that contains either only x or only y.

X is T_1 space or satisfies the T_1 axiom if for every pair of distinct points $x, y \in X$ $\exists$ nbds $U_x \subset X$ of x and $U_y \subset X$ of y

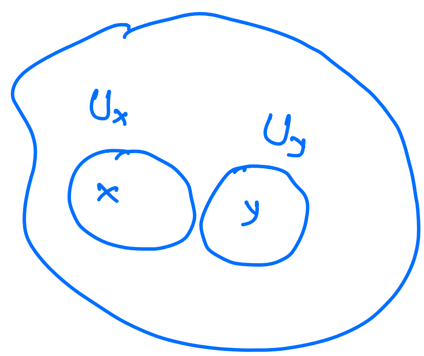
s.t. $x \notin U_y$ and $y \notin U_x$.



every T_1 space is T_0.

X is a T_2 space = Hausdorff space if $\forall$ $x, y \in X$ $\exists$ $U_x \ni x$, $U_y \ni y$ s.t.

$U_x \cap U_y = \phi$.



One can ask, can we separate points using continuous functions?