Lecture 6

Recall :-

* Product topology and box topology on $\prod_{\alpha\in I} X_\alpha$.

* Countability Axioms - 1^{st} countable if $\forall x \in X$ $\exists$ a countable nbd basis.

  2^{nd} countable if X has a countable basis.

* $f : X \longrightarrow Y$ and let X be 1^{st} countable. Then f continuous $\Longleftrightarrow$ f sequentially continuous.

Compactness

X top. space, $A \subset X$, an open cover of A is a collection of open sets $\{U_\alpha\}_{\alpha\in I}$ if

$$A \subset \bigcup_{\alpha\in I} U_\alpha .$$

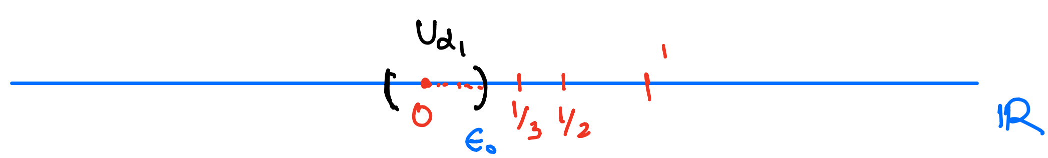
Def^n $A \subset X$ is compact if every open cover of A admits a finite subcover, i.e. if $\{U_\alpha\}_{\alpha\in I}$ is an open cover of A then $\exists$ $\alpha_1, \dots, \alpha_N \in I$ s.t

$$A \subset \bigcup_{i=1}^{n} U_{\alpha_i}.$$

To show that a set is not compact, just need to produce one open cover which can never admit a finite subcover.

Examples:- i) $\mathbb{R}$, standard topology is not compact. $\{(-n,n) \mid n \in \mathbb{N}\}$ doesn't admit a finite subcover.

ii) $A = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is compact.



Suppose $\{U_\alpha\}_{\alpha \in I}$ is an open cover for A.

$\Rightarrow$ $\exists$ some U_{α_1} s.t. $0 \in U_{\alpha_1}$.

$\Rightarrow$ there are only finitely many $n \in \mathbb{N}$ s.t. $1/n \notin U_{\alpha_1}$, say $n_1, n_2, \ldots, n_K$.

Choose the open sets from $\{U_\alpha\}_{\alpha \in I}$ containing $\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_K}$ say $U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_{K+1}}$

then $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_{k+1}}\}$ is a finite subcover for A.

iii) Any finite set in any Topological space is compact.

iv) $(0, 1] \subset \mathbb{R}$ is not compact.

$\{(1/n, 1] \mid n \in \mathbb{N}\}$ is an open which can't admit a finite sub cover.

<u>Remark</u> :- Compact subset of a topological space is a compact space w.r.t. the subspace topology.

v) compact sets of a discrete metric space?

$\underset{\text{discrete top.}}{A \subset X}$ is compact $\iff$ A is finite.

<u>Exer</u> what are compact subsets of $\mathbb{R}$ w/ cofinite topology?

vi) Heine – Borel thm :- $A \subset \mathbb{R}^n$ is compact $\iff$ A is closed and bounded.

There are examples of closed and bounded sets which are not compact.

$\mathcal{H}$ real inner product space is a Hilbert space if $\mathcal{H}$ is complete, i.e, every Cauchy sequence in $\mathcal{H}$ converges in $\mathcal{H}$.

$$d(x,y) = \sqrt{\langle x-y, x-y \rangle}$$

$\mathcal{H}$ ∞-dimensional

$\overline{B_1(0)} = \{ x \in \mathcal{H} \mid \langle x, x \rangle \leq 1 \}$ is closed and bounded. But $\overline{B_1(0)}$ is NOT compact.

$\{e_1, e_2, e_3, \dots\}$ orthonormal set in $\mathcal{H}$, i.e.,

$\forall i$, $\langle e_i, e_i \rangle = 1$ and $\langle e_i, e_j \rangle = 0$ for $i \neq j$

$d(e_i, e_j) = \sqrt{2}$ when $i \neq j$

$\Rightarrow$ $r < \frac{\sqrt{2}}{2}$ then no ball of radius r can contain more than one of the vectors in $\{e_1, e_2, \dots\}$.

$\Rightarrow$ the open cover $\{ B_r(x) \mid x \in \mathcal{H}, r \text{ is as above} \}$ can never admit a finite subcover. $\boxtimes$

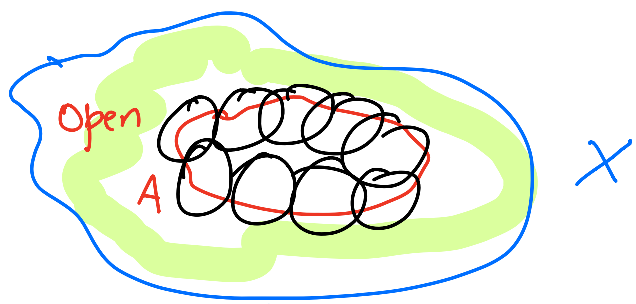
Theorem :- X is a compact space. and let $A \subset X$ be closed. Then A is compact.

Proof :- Suppose we have an open cover of A. $\{U_\alpha\}_{\alpha \in I}$. i.e, $A \subset \bigcup_{\alpha \in I} U_\alpha$.

$\because$ A is closed $\Rightarrow$ $X \setminus A$ is open

$\Rightarrow$ $\{U_\alpha\}_{\alpha \in I} \cup \{X \setminus A\}$ is an open cover for X.

$\Rightarrow$ it admits a finite subcover (as X is compact)

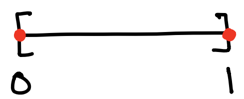


$\Rightarrow$ $\{U_\alpha\}_{\alpha \in I}$ admits a finite subcover for A. $\boxtimes$

Exer. Compact space of a Hausdorff space is closed.

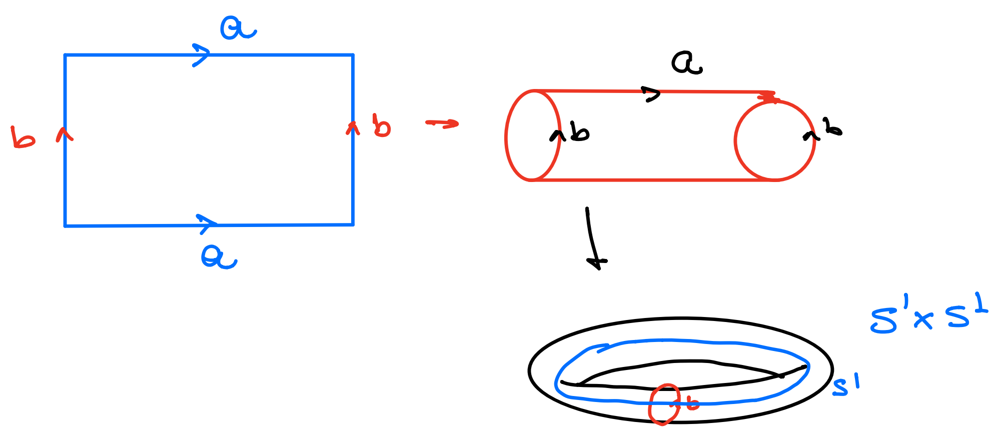
Compactness is a Topological property, i.e, compact-ness is preserved under continuous maps (homeomorphisms), i.e, if $f: X \to Y$ is continuous and $A \subset X$ is compact in X $\Rightarrow$ $f(A)$ is compact in Y.

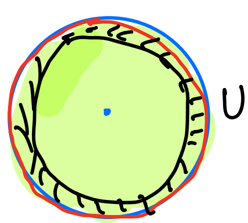
Quotient Topology



$\rightsquigarrow$ circle S^1

$\mathbb{R}/\mathbb{Z} \rightsquigarrow$ define an equivalence relation on $\mathbb{R}$ as $x \sim y$ if $x - y \in \mathbb{Z}$. $\rightsquigarrow S^1$





identifying the boundary to a point

Def^n Let X and Y be topological spaces.

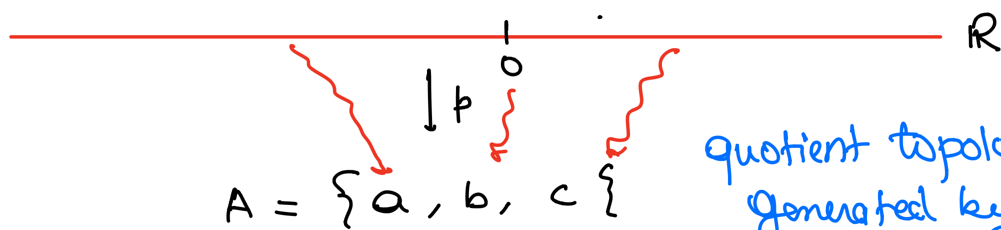
Let $p : X \longrightarrow Y$ be a surjective map. p is a quotient map if $U \subset Y$ is open $\iff p^{-1}(U)$ is open in X.

Defn Let A be a set and let $p: X \to A$ be a surjective map. The strongest topology on A w.r.t. which p is a quotient map is the quotient topology on A induced by p.

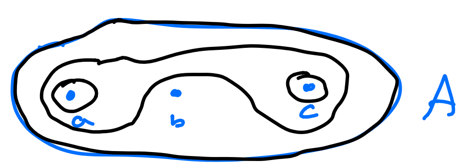
The quotient topology on A can be described as :-

$B \subset A$ open $\iff$ $p^{-1}(B)$ is open in X.

Check that this indeed gives a topology.



quotient topology generated by p on A



Defn Let X be a top. space and $\sim$ is an equivalence relation on X and let X^* be a partition of X. Consider

$$p : X \to X^* \text{ as the surjective map}$$
$$x \longmapsto [x]$$

In the quotient topology induced by p, X^* is called the quotient space of X.

Remark :- Quotient space of a metric space need not be a metric space.