Lecture 5

- PSet 2 on course webpage/moodle and is due on 04/05/2021.
- Will discuss PSet 1 today in the problem sessi--ons.

Recall :- Suppose $(X_i, \mathcal{T}_i)$ $i=1,2$, the product topology on $X_1 \times X_2$ is the weakest topology s.t.

$$\pi_i : X_1 \times X_2 \longrightarrow X_i \text{ is continuous.}$$

If $U_1 \in \mathcal{T}_1$ in X_1 $\Rightarrow$ $\pi_1^{-1}(U_1)$ should be open in $X_1 \times X_2$.

$$\pi_1^{-1}(U_1) = U_1 \times X_2 \text{ is open in } X_1 \times X_2$$

$U_2 \in \mathcal{T}_2 \Rightarrow \pi_2^{-1}(U_2)$ open in $X_1 \times X_2$

$\Rightarrow$ $\pi_2^{-1}(U_2) = X_1 \times U_2$ open in $X_1 \times X_2$.

$\Rightarrow$ $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) = U_1 \times U_2$ must be open in $X_1 \times X_2$.

$\Rightarrow$ $\{ \pi_1^{-1}(U_1) \mid U_1 \in \mathcal{T}_1 \} \cup \{ \pi_2^{-1}(U_2) \mid U_2 \in \mathcal{T}_2 \}$

forms a subbase for the product topology $\mathcal{T}$ on $X_1 \times X_2$.

Suppose $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$ collection of topological spaces

$\underset{\sim}{\text{any}}$ set (could be uncountable).

$$\prod_{\alpha \in I} X_\alpha = \left\{ \text{functions } f : I \to \bigcup_{\alpha \in I} X_\alpha \;\middle|\; \alpha \mapsto x_\alpha \text{ s.t. } x_\alpha \in X_\alpha \;\forall \alpha \in I \right\}$$

$\{x_\alpha\}_{\alpha \in I}$ or $(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$

each $x_\alpha \in X_\alpha \rightsquigarrow \alpha^{th}$ coordinate in $\prod_{\alpha \in I} X_\alpha$.

$$X_1 \times X_2 = \left\{ f : \{1,2\} \to X_1 \cup X_2 \;\middle|\; \begin{array}{l} f(1) \mapsto X_1 \\ f(2) \mapsto X_2 \end{array} \right)$$

(x_1, x_2)

Def^n The product topology on $\prod_{\alpha \in I} X_\alpha$ is the weakest topology s.t.

$$\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha : \{x_\beta\}_{\beta \in I} \mapsto x_\alpha \in X_\alpha$$

is continuous $\forall \alpha \in I$.

$\Rightarrow$ if $U_\alpha \in \tau_\alpha \Rightarrow \pi_\alpha^{-1}(U_\alpha)$ should be open in $\prod_{\beta \in I} X_\beta$ $\forall \alpha \in I$.

$\Rightarrow \quad \{\pi_\alpha^{-1}(U_\alpha)\}_{\alpha\in I}$ form a subbase in the product topology.

$$\pi_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\substack{\beta\in I \\ \beta\neq\alpha}} X_\beta .$$

Any open set in the product topology on $\prod_{\beta\in I} X_\beta$ must be written as a union of finite intersections of $\pi_\alpha^{-1}(U_\alpha)$, $\alpha\in I$.

$\Rightarrow$ A base for the product topology on $\prod_{\alpha\in I} X_\alpha$ is a collection of subsets of the form

$\prod_{\alpha\in I} U_\alpha$ s.t. $U_\alpha \subset X_\alpha$ is open $\forall \alpha\in I$ and $U_\alpha \neq X_\alpha$ for only finitely many $\alpha\in I$.

an arbitrary open set might look like

$$\underbrace{U_{\alpha_1}\times U_{\alpha_2}\times\cdots\times U_{\alpha_n}}_{U_{\alpha_i}\in \mathcal{T}_{\alpha_i}} \times \prod_{\substack{\beta\in I \\ \beta\neq\alpha_1,\alpha_2,\ldots,\alpha_n}} X_\beta .$$

Exer :- i) $\{x_\alpha^n\}_{\alpha\in I}$ is a sequence in $\prod_{\alpha\in I} X_\alpha \to \{x_\alpha\}_{\alpha\in I}$ w/ product topology $\Longleftrightarrow$ the individual sequence $x_\alpha^n \to x_\alpha$ in X_α.

ii) $f: Y \to \prod_{\alpha\in I} X_\alpha$ is continuous $\Longleftrightarrow$ $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous $\forall \alpha\in I$.

Box Topology

$\{(X_\alpha, \tau_\alpha)\}_{\alpha\in I}$ topological spaces.

$\prod_{\alpha\in I} X_\alpha$, box topology is generated by the basis

$$\left\{ \prod_{\alpha\in I} U_\alpha \;\middle|\; U_\alpha \in \tau_\alpha \;\; \forall \alpha\in I \right\}$$

Ex. i) compare the box topology w/ product topology on $\prod_{\alpha\in I} X_\alpha$.

ii) What does convergence of sequence $\{x_\alpha^n\} \in \prod_{\alpha\in I} X_\alpha$ look like?

- countability axioms ✓
- compactness
- Quotient Topology / connectedness & path-connectedness.

For metric spaces, $f: X \to Y$ is continuous if

i) $f^{-1}(U)$ is open in X whenever U is open in Y.

$$\Updownarrow$$

ii) $x_n \to x$ in $X \Rightarrow f(x_n) \to f(x)$ in Y.

<u>remarks</u>:- i) is true for arbitrary topological spaces.

convergence makes sense for " —— " —— "

i) $\Rightarrow$ ii) holds for " ——— "

ii) $\Rightarrow$ i) Needed the metric.

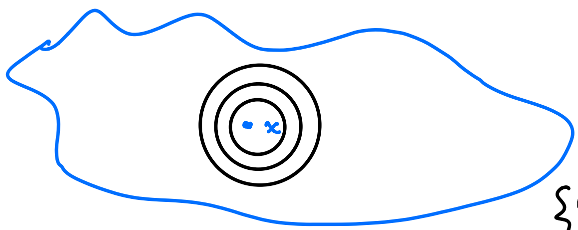
The criteria ii) is known as sequential continuity.

<u>Defn</u> X is a topological space, $x \in X$.

A neighbourhood base of x is a collection $\mathcal{B}$ of neighbourhoods of x s.t every nbd of x contains some $U \in \mathcal{B}$.

↓ generalizing

$B_{1/n}(x)$ for metric spaces.



X metric

$\{B_{1/n}(x)\}$ is countable.

Def^n A space X is called first countable if every point $x \in X$ has a countable neighbourhood base.

X is called second countable if its topology has a countable base.

2^{nd} countable $\Longrightarrow$ 1^{st} countable

$\Longleftarrow$ NOT true.

Ques Is every metric space 1^{st} countable?

Yes $\rightsquigarrow$ $B_{\frac{1}{n}}(x)$

Ques. " ——————————" 2^{nd} countable?

NO $\rightarrow$ X uncountable w/ discrete topology.

$\{x\}$ open set.

X, discrete topology is 2^{nd} countable

$\Updownarrow$

X is countable.

Example of Topological space which is not 1st countable

($\mathbb{R}$, cofinite topology, cocountable topology) is NOT 1st countable.

$x \in \mathbb{R}$ suppose there is a countable nbd basis

$\{U_i \mid i \in \mathbb{N}\} \Rightarrow U_i^c$ is countable $\forall i$

$U = \bigcap U_i$

$$\Rightarrow U^c = \left(\bigcap_{i \in \mathbb{N}} U_i\right)^c = \bigcup U_i^c$$ is also countable.

consider $U \setminus \{y\}$ $(y \neq x)$ has countable complement $\Rightarrow U \setminus \{y\}$ is open and $x \in U \setminus \{y\}$

But it can never contain any $\{U_i \mid i \in \mathbb{N}\}$.

$U_i \not\subseteq U \setminus \{y\}$.

Theorem X, Y are topological spaces, X is 1st countable, then every sequentially continuous map $f : X \to Y$ is also continuous.

Proof requires the following Lemma.

Lemma:- X is 1st countable, $A \subset X$. A is NOT open $\Longleftrightarrow \exists\ x \in A$ and a sequence $x_n \in X \setminus A$ s.t. $x_n \to x$.

Proof of the Lemma

If $A \subset X$ is open $\forall x \in A$, $x_n \in X$ s.t. $x_n \to x$ we can't have $x_n \in X \setminus A$ $\forall n$ b/c A itself is a nbd of x.

Suppose A is not open in X. $\Rightarrow$ $\exists\, x \in A$ s.t. no nbd $x \in U$ of X is contained in A.

Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable nbd basis for x.

WLOG, assume $\{U_i\}_{i \in \mathbb{N}}$ forms a nested sequence of nbd.

$$X \supset U_1 \supset U_2 \supset U_3 \supset \cdots \ni x$$

$\because$ U_i is a nbd of x, none of these U_i can be contained in A $\Rightarrow$ $\exists$ a sequence of points

$$x_n \in U_n \quad \text{s.t.} \quad x_n \notin A.$$

(x_n) in X, $(x_n) \to x$ as every nbd $x \in V \subset X$ must contain U_i for some i b/c of the def^n of a nbd basis.

$$\Rightarrow \forall\, j > i,\ x_j \in V \Rightarrow (x_n) \to x$$

$\therefore$ we have a sequence $(x_n) \in X \setminus A$ s.t. $x_n \to x$ if A is not open.

$\boxtimes$