Lecture 4

Recall :- (X, τ) is a topological space if $\tau \subseteq P(X)$ (powerset of X) and satisfy the following:-

i) $X, \phi \in \tau$

ii) $\{U_i\}_{i \in I} \in \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau$. (Arbitrary union of open sets is an open set).

iii) $U_1, U_2, \dots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{n} U_i \in \tau$. (finite inter-section of open sets is an open set.)

elements of τ are called open sets.

→ $A \subset (X, \tau)$ is closed set if $A^c = X \setminus A \in \tau$.

→ $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is continuous if $\forall\ U \in \tau_Y,\ f^{-1}(U) \in \tau_X$.

f is continuous if $\forall$ closed set V in Y $f^{-1}(V)$ is closed in X.

→ $x_n \longrightarrow x \in X$ if $\forall\ U \in \tau_X,\ x \in U\ \exists\ n \in \mathbb{N}$ st $\forall\ m \geq n,\ x_m \in U$.

Examples i) $\mathcal{T} = \{X, \phi\}$ is a topology on X.

Trivial topology on X.

ii) $X = \{a, b, c\}$

$\mathcal{T}_1 = \{X, \phi\}$, $\mathcal{T}_2 = \{X, \phi, \{a,b\}, \{b\}, \{b,c\}\}$

$\mathcal{T}_3 = \{$ every subset of $X\}$

$\mathcal{T}_4 = \{X, \Phi, \{a,b\}, \{c\}\}$

non-example

$T_1 = \{X, \Phi, \{a\}, \{b\}\}$

$T_2 = \{X, \Phi, \{a,b\}, \{b,c\}\}$

NOT topologies on X.

Def $(X, \mathcal{T})$ topological space $\mathcal{B} \subset \mathcal{T}$ subcollection (every element of $\mathcal{B}$ is an open set).

1. $\mathcal{B}$ is a basis for $\mathcal{T}$ if every open set $U \in \mathcal{T}$ is a union of sets in $\mathcal{B}$, i.e,

$$U = \bigcup_{\alpha \in I} U_\alpha \quad , \quad U_\alpha \in \mathcal{B}.$$

2. $\mathcal{B}$ is a subbasis for $\mathcal{T}$ if every open set U is a union of finite intersections of sets in

$\mathcal{B}$, i.e.

$$U = \bigcup_{\alpha \in I} U_\alpha \quad \text{where}$$

$$U_\alpha = U_\alpha^1 \cap U_\alpha^2 \cap \cdots \cap U_\alpha^n, \quad \{U_\alpha^i\}_{i=1}^n \in \mathcal{B}.$$

Every basis is a subbasis.

Ex. contd.

$(\mathbb{R}, |\cdot|)$

basis $\mathcal{B} = \{(a,b) \mid -\infty \le a < b \le \infty\}$

subbasis $\mathcal{B}' = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{(a, \infty)\}$

$\mathcal{B}'$ is NOT a basis for usual topology on $\mathbb{R}$.

(Check).

→ (X, d) metric space.

$\mathcal{B} = \{B_r(x) \mid x \in X, r > 0\}$ is a basis for (X, d).

→ X is with discrete topology. $= \mathcal{T} = \{\text{subsets of } X\}$

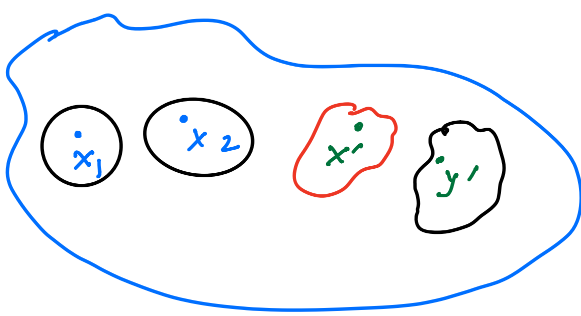
$\mathcal{B} = \{\{x\} \mid x \in X\}$ is a basis.

Def^n (X, τ) is metrizable if τ is the topology induced by a metric.

Not every top. space is metrizable.

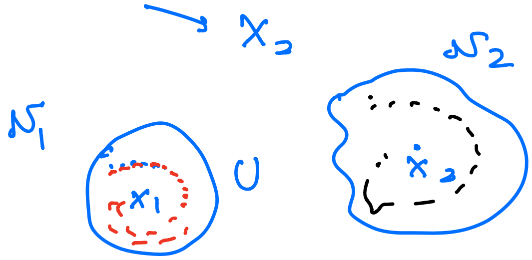
Proposition :- A sequence $x_n \in X$ has a unique limit if (X, τ) is metrizable.

Proof :- Hausdorff spaces $x_1, x_2 \in X$ then $\exists$ open sets $U, V \in \tau$ s.t. $x_1 \in U$, $x_2 \in V$ and $U \cap V = \phi$.



Every metric space is a Hausdorff space.

Suppose $x_n \to x_1$, $\to x_2$ $x_1 \neq x_2$



$U \cap V = \phi$

$N = \max\{N_1, N_2\}$ $\underline{x_m}$, $m > N$

contradiction $\Rightarrow x_1 = x_2$.

$\square$

* (X, τ) where τ is the trivial topology is NOT metrizable.

$(x_n) \in X = \{x_1, x_2, x_3 \dots \}$

any sequence in X converges to every point.

$\Rightarrow$ (X, τ) is NOT metrizable.

Ex. X τ cofinite topology

$$\tau = \{ U \subset X \mid X \setminus U \text{ is finite} \text{ or } X \}.$$

Check - (X, τ) is a topological space.

- $f : X \to \mathbb{R}$ what are the continuous function f here?

when X has the trivial topology.

When is $f : \mathbb{R} \to X$ continuous?

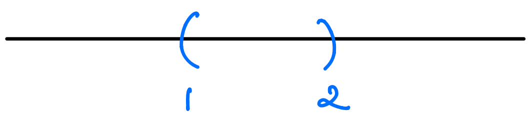
X τ_1 τ_2 are topologies on X.

$\tau_1 \subset \tau_2$ iff every open set in (X, τ_1) is also an open set in (X, τ_2).

In this case, we say τ_2 is stronger/finer topology than τ_1 and τ_1 is weaker/coarser topology than τ_2.

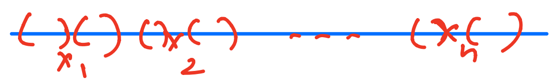
Remark :- (i) trivial topology is <u>weakest</u> on X
(ii) discrete topology is <u>strongest.</u> on X.

$(\mathbb{R}, |\cdot|)$ usual topology $\supset$ $\mathbb{R}$, cofinite topology
& finer



open in usual but not in cofinite:

U open
$\Rightarrow \mathbb{R} - U = \{x_1, \dots x_n\}$



Given τ_1 and τ_2 on X they might <u>NOT</u> be comparable.

$(\mathbb{R}, \text{usual})$ NOT comparable. $(\mathbb{R}, \text{cocountable topology})$
check!
U is an open set if $\mathbb{R} \setminus U$ is countable.

Subspace Topology, Product Topology (Box topology) & Quotient topology.

<u>Subspace Topology</u>

(X, τ) is a topological space. $A \subset X$. The subspace topology on A is $\mathcal{B}$ - basis for X

$$\tau_A = \{ U \cap A \mid U \in \tau \}.$$ satisfies the axioms of a top. space.

$\mathcal{B}_A = \{ U \cap A \mid U \in \mathcal{B} \}$ basis for (A, τ_A).

* $A \xrightarrow{i} X$ inclusion map.
  subspace topology on A is the weakest topology on A s.t. i is a continuous map.

* (X, d) metric spae, $A \subset X$
  then the topology generated by d_A = subspace topology according to the above defn.

Every subspace of a metrizable space is metrizable.

Product Topology

(X_1, τ_1) and (X_2, τ_2) are topological spaces.

The product topology τ on $X_1 \times X_2$ is

$$\tau = \{ U \times V \mid U \in \tau_1, V \in \tau_2 \}.$$

($\mathcal{B}_1$, $\mathcal{B}_2$)

basis for $(X_1 \times X_2, \tau)$ is

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{B}_1, V \in \mathcal{B}_2 \}.$$

$\pi_1 : X_1 \times X_2 \longrightarrow X_1 \quad (x_1, x_2) \xmapsto{\pi_1} x_1$

$\pi_2 : X_1 \times X_2 \longrightarrow X_2 \quad (x_1, x_2) \xmapsto{\pi_2} x_2$

projections maps.

τ is the weakest topology on $X_1 \times X_2$ s.t. π_1 and π_2 are continuous.