Lecture 3

* Pset 1 has been posted on the course webpage and moodle.
  Due Date :- 27/04/2021 at 3:15 PM.

* Problem session today $\rightsquigarrow$ open office hour.

Recall :- - metric spaces

- Open balls, open sets & neighbourhoods
- convergence of sequence in metric spaces
- Continuous function & homeomorphism

(U open $\Rightarrow$ $f^{-1}(U)$ is open)   (f continuous, bijection

$X \cong Y$ f^{-1} continuous)

Lemma :- $(\mathbb{R}^n, d_E) \cong (B_r(0), d_E)$.

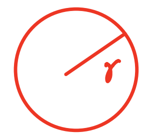
Cor :- All open balls in $(\mathbb{R}^n, d_E)$ are homeomorphic to each other and $\therefore$ to $(\mathbb{R}^n, d_E)$.

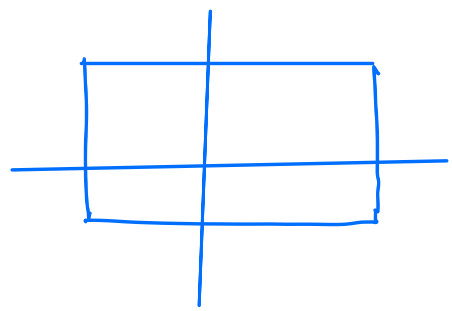
$$f : B_r(0) \longrightarrow \mathbb{R}^n \qquad \text{bijective}$$

$$x \longmapsto \frac{x}{r - \|x\|_2} \qquad \text{continuous}$$

$$f^{-1} = g : \mathbb{R}^n \longrightarrow B_r(0) \qquad \text{continuous.}$$

$$y \longmapsto \frac{ry}{1+\|y\|_2}$$

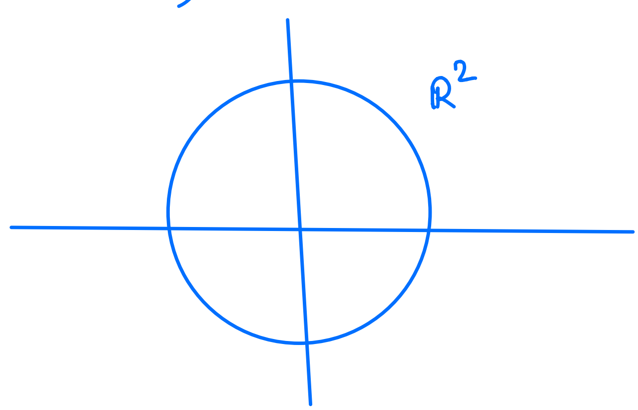




Def: In (X,d), a subset $A\subset X$ is said to be bounded if $\exists\ M\geq 0$ s.t.

$$d(a,b)\leq M \qquad \forall\ a,b\in A.$$

$(B_r(0), d_E)$ is bounded. $(\mathbb{R}^n, d_E)$ is NOT bounded.



Def^n A topological property is the one which is preserved by homeomorphism.

Boundedness is NOT a topological property.

(X, d) and (X, d')

Def^n Two metrics d and d' on X are called (topologically) equivalent if $id : (X, d) \to (X, d')$, $x \mapsto x$, is a homeomorphism.

Exer :- $id : (X, d) \to (X, d')$ is a homeomorphism

$\Longleftrightarrow$

$x_n \to x$ in $(X, d) \Longleftrightarrow x_n \to x$ in (X, d')

$\Longleftrightarrow$

open sets in $(X, d) \Longleftrightarrow$ open set in (X, d').

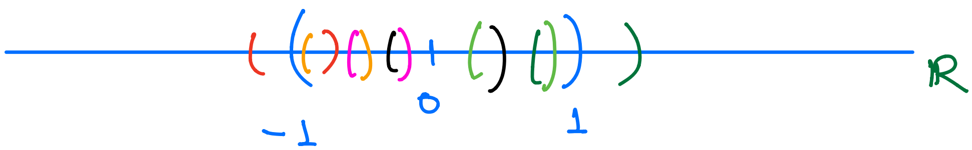
Compactness

Suppose I is an index set.

A collection $\{U_\alpha\}_{\alpha \in I}$ of open sets of X ($U_\alpha \subset X$ $\forall \alpha \in I$) is an open cover of $A \subset X$ if

$$A \subset \bigcup_{\alpha \in I} U_\alpha .$$

Remark This def^n makes sense for arbitrary "topological spaces".

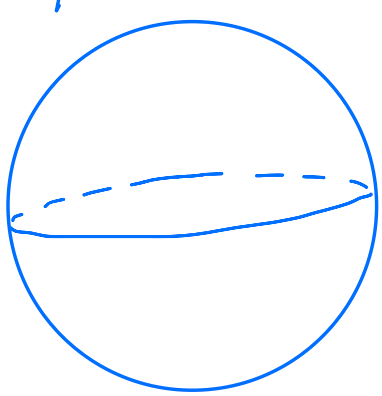


Defn A subset $A \subset (X, d)$ is compact if either of the equivalent condition holds:-

(a) Every open cover $\{U_\alpha\}_{\alpha \in I}$ of A has a finite subcover, i.e. there is a finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset I$ s.t

$$A \subset \bigcup_{i=1}^{n} U_{\alpha_i}.$$

(b) Every sequence $x_n \in A$ has a convergent subsequen-ce w/ limit in A.



$S^n \subset \mathbb{R}^{n+1}$ is compact in $\mathbb{R}^{n+1}$!

Theorem:- Compactness is a topological property, i.e. $f: X \to Y$ is continuous and suppose $A \subset X$ is compact then so is $f(A) \subset Y$.

Proof- $A \subset X$ is compact.

Want:- $f(A) \subset Y$ is compact.

Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover for $f(A)$.

$$f(A) \subset \bigcup_{\alpha \in I} V_\alpha$$

Look at $f^{-1}(V_\alpha)$ - open in X as f is continuous.

$\{f^{-1}(V_\alpha)\}_{\alpha \in I}$ is an open cover for A.

$\Rightarrow$ $\exists \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset I$ s.t.

$$A \subset \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}).$$

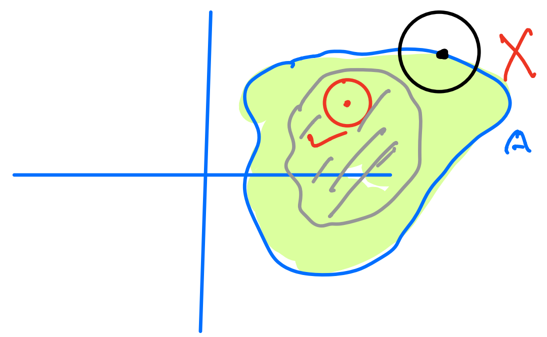
$$\Rightarrow \quad f(A) \subset \bigcup_{i=1}^{n} V_{\alpha_i}. \qquad \blacksquare$$

Topological spaces

Def^n:- X is a metric (or topological) space.

(a) interior of a subset $A \subset X$ is the set

$$\overset{\circ}{A} = \{ x \in A \mid \exists \text{ a nbd } U \text{ of } x \text{ in } X \text{ w/ } U \subset A \}$$



$\odot$ not a cluster point.

(b) closure of $A \subset X$ is the set

$$\bar{A} = \{ x \in X \mid \text{every nbd of } x \text{ in } X \text{ intersects } A \}$$

$x \in \bar{A}$ is called a cluster point.

Exer:- $A \subset X$

$\overset{\circ}{A}$ is the largest open subset of X that is contained in A, i.e.

$$\overset{\circ}{A} = \bigcup_{U \subset X \text{ open and } U \subset A} U$$

$\bar{A}$ is the smallest closed subset of X that contains A, i.e.

$$\bar{A} = \bigcap_{V \subset X \text{ closed},\ V \supset A} V$$

Note:- Arbitrary union of open sets is open in a metric space.

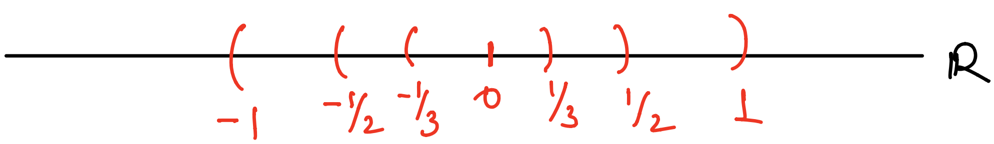
$$x \in \bigcup_{\alpha \in I} V_\alpha \implies x \in \underbrace{V_{\alpha_0}}_{\text{open}} \implies \exists \text{ some } B_r(x) \subset V_{\alpha_0}$$

$$\implies B_r(x) \subset \bigcup_{\alpha \in I} V_\alpha$$

$\Updownarrow$

Arbitrary intersection of closed sets is closed.

$$\left(\bigcap_{\alpha \in I} V_\alpha \right)' = \bigcup_{\alpha \in I} \underbrace{V_\alpha'}_{\text{open}}$$



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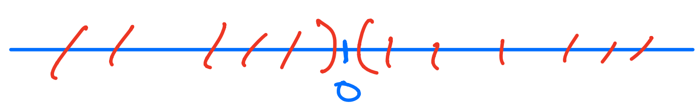
$$\bigcap \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\} \text{ is closed in } \mathbb{R}$$



Def^n:- A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following:-

i) $\phi, X \in \mathcal{T}$.

ii) If $\{U_\alpha\}_{\alpha \in I} \in \mathcal{T} \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

iii) If $\{U_{\alpha_i}\}_{i=1}^{n} \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} U_{\alpha_i} \in \mathcal{T}$.

Def.:- Elements of $\mathcal{T}$ (subsets of X) are called open sets in X.