Lecture 28

Recall:-

Relative homology

(X, A) , $A \subset X$ ⇝ pair of spaces

$f : (X, A) \longrightarrow (Y, B)$ map of pair.

$f(A) \subset B$.

f, g map of pairs. ⇝ homotopic if $\exists$

$H : I \times X \longrightarrow Y$ s.t. $\forall\, s \in I$ $H(s, \cdot) : (X, A) \to (Y, B)$

is a map of pair.

If $\sigma \in C_n(A; G) \implies \sigma : \Delta^n \longrightarrow A \subset X$

$\implies \sigma \in C_n(X; G)$

$\implies C_n(A; G) \leq C_n(X; G)$

$\partial : C_n(X, G) \longrightarrow C_{n-1}(X; G)$ descends to the boundary

$\partial : C_n(A, G) \longrightarrow C_{n-1}(A, G) \implies$

$$\underbrace{C_n(X, A; G)}_{\text{relative singular n-chain group.}} = C_n(X; G) / C_n(A; G)$$

$(C_*(X, A; G), \partial)$ relative singular chain complex
$(\partial^2 = 0)$

↓ Homology groups of this complex are called relative singular hom. groups of the pair (X, A).

$A = \phi$ the rel. sing. hom. gp. $(X, \phi) \rightsquigarrow$ absolute hom. groups.

If $f : (X, A) \longrightarrow (Y, B)$ is a map of pair then the absolute chain map $f_* : C_*(X; G) \to C_*(Y; G)$ sends $C_*(A; G)$ into $C_*(B; G) \Rightarrow$ we get a chain map

$$f_* : C_*(X, A; G) \longrightarrow C_*(Y, B; G)$$

$$\rightsquigarrow$$

group homomorphisms $f_* : H_n(X, A; G) \longrightarrow H_n(Y, B; G)$.

$$(f \circ g)_* = f_* \circ g_*$$

$Id : (X, A) \to (X, A) \rightsquigarrow (id)_* = id : H_n(X, A; G)$ ⟲

$$C_n(X, A; G) = \frac{C_n(X; G)}{C_n(A; G)}$$

$\cup$

$c \rightsquigarrow$ we can view this as a n-chain in X, i.e., as an element of $C_n(X; G)$

If $a, b \in C_n(X; G) \rightsquigarrow a, b \in C_n(X, A; G)$ then $a = b$ in $C_n(X, A; G) \Longleftrightarrow a - b \in C_n(A; G)$

$c \in C_n(X; G)$ is called a relative cycle if the correspo-

-nding element $c \in C_n(X,A;G)$ is a cycle $\Longrightarrow$

$\partial c = 0$ in $C_{n-1}(X;A;G) \Longrightarrow \partial c \in C_{n-1}(A;G)$

A relative cycle need NOT be an absolute cycle.
But an absolute cycle is always a rel. cycle.

$[b] = [c]$ in $H_n(X,A;G)$

$\Updownarrow$

$b - c = a + \partial x$ for some $a \in C_n(A;G)$
$x \in C_{n+1}(X;G)$

$\left(H_n(X;G) = \dfrac{Z_n(X;G)}{B_n(X;G)} \right)$

$X \supset A$ $\qquad (X,A)$

$H_*(X;G)$ $\quad H_*(A;G)$ $\quad H_*(X,A;G)$

$i : A \hookrightarrow X$
$j : (X,\phi) = X \longrightarrow (X,A)$ } inclusions.

} induces maps at the chain complex level

We consider the following sequence of chain maps.

$$0 \longrightarrow C_*(A,G) \xrightarrow{i_*} C_*(X,G) \xrightarrow{j_*} C_*(X,A,G) \longrightarrow 0 \quad \text{——— } (1)$$

$\because$ i is the inclusion map $\Rightarrow$ i_* is injective.

j_* is surjective

$$j_* : C_*(X,G) \longrightarrow \frac{C_*(X,G)}{C_*(A,G)} \quad \text{projection map} \Rightarrow \text{it is surjective.}$$

$\because$ $\ker j_*$ are all the n-chains in X which are actually n-chain in A $= im(i_*)$

$$\ker j_* = im(i_*) \qquad \text{———} \qquad (2)$$

$$\cdots \longrightarrow 0 \longrightarrow C_{n-1}(A,G) \xrightarrow{i_*} C_{n-1}(X,G) \xrightarrow{j_*} C_{n-1}(X,A,G) \longrightarrow 0$$

$$im = 0 = \ker(i_*) = 0 \qquad\qquad im\, j_* = C_{n-1}(X,A,G)$$

$$0 \longleftarrow C_n(X,A;G) \xleftarrow{j_*} C_n(X,G) \xleftarrow{i_*} C_n(A,G) \longleftarrow 0$$

$$\vdots$$

In general, a sequence of abelian groups w/ homo.

$$\cdots \longrightarrow A_{n-2} \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots$$

is exact iff $\ker(f_{i+1}) = Im(f_i)$ $\forall$ i.

The special case

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is called a short exact sequence. Exactness tells that f_2 is surjec. f_1 is inj and $im(f_1) = \ker f_2$.

case in (1) is called a short exact sequence of chain maps

Theorem (Short exact sequence gives rise to long exact sequence)

Let (A_*, ∂^A), (B_*, ∂^B) and (C_*, ∂^C) be chain complexes and suppose

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

is a short exact sequence of chain complexes. Then $\exists$ a natural homomorphism $\partial_* : H_n(C_*, \partial^C) \longrightarrow H_{n-1}(A_*, \partial^A)$ $\forall\, n \in \mathbb{Z}$ s.t. the sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_{n+1}(A_*, \partial^A) \arrow[r, "f_*"] & H_{n+1}(B_*, \partial^B) \arrow[r, "g_*"] & H_{n+1}(C_*, \partial^C) \arrow[d, "\partial_*"] \\
H_{n-1}(A_*, \partial^C) \arrow[d] & H_n(C_*, \partial^C) \arrow[l, "\partial_*"'] & H_n(B_*, \partial^B) \arrow[l, "g_*"'] & H_n(A_*, \partial^C) \arrow[l, "f_*"'] \\
\vdots & & &
\end{tikzcd} \qquad (2)$$

is exact.

"Sketch of the proof"

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial^A"] & \vdots \arrow[d, "\partial^B"] & \vdots \arrow[d, "\partial^C"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "f"] \arrow[d, "\partial^A"] & B_{n+1} \arrow[r, "g"] \arrow[d, "\partial^B"] & C_{n+1} \arrow[r] \arrow[d, "\partial^C"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "\partial^A"] & B_n \arrow[r, "g"] \arrow[d, "\partial^B"] & C_n \arrow[r] \arrow[d, "\partial^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f"] \arrow[d, "\partial^A"] & B_{n-1} \arrow[r, "g"] \arrow[d, "\partial^B"] & C_{n-1} \arrow[r] \arrow[d, "\partial^C"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "f"] \arrow[d, "\partial^A"] & B_{n-2} \arrow[r, "g"] \arrow[d, "\partial^B"] & C_{n-2} \arrow[r] \arrow[d, "\partial^C"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

<u>Want:-</u>

$$\partial_* : H_n((C_*, \partial^C)) \longrightarrow H_{n-1}(A_*, \partial^A)$$

$$\mathrel{\rotatebox[origin=c]{90}{$\in$}}$$

$$[c] \implies c \in C_n \text{ is a cycle} \implies \partial^C c = 0$$

$\because$ $g : B_n \to C_n$ is surjective due to exactness

$\implies$ $c = g(b)$ for some $b \in B_n$.

By g being a chain map

$$0 = \partial^C c = \partial^C g(b) = g(\partial^B b)$$

$\implies$ $\partial^B b \in \ker g \subset B_{n-1}$.

By exactness, we know that $\ker g = \operatorname{Im} f$

$\Rightarrow \quad \partial^B b = f(a)$ for some $a \in A_{n-1}$.

a is unique as f is injective.

By commutativity

$$f(\partial^A a) = \partial^B(f(a)) = \partial^B(\partial^B b) = 0$$

$\Rightarrow$ $f(\partial^A a) = 0$ but f is injective $\Rightarrow \partial^A a = 0$

$\Rightarrow \quad a \in A_{n-1}$ is indeed a cycle.

We can now define $\partial_* : H_n(C_*, \partial^C) \to H_{n-1}(C_*, \partial^A)$

$$\partial_* [c] = [a] \in H_{n-1}(A, \partial^A) \quad \text{—— } (3)$$

There were two choices involved in this procedure.

1) representative of $[c]$

2) $b \in g^{-1}(c)$

exer. 1) ∂_* is indeed independent of these choices.

2) ∂_* is a homomorphism.

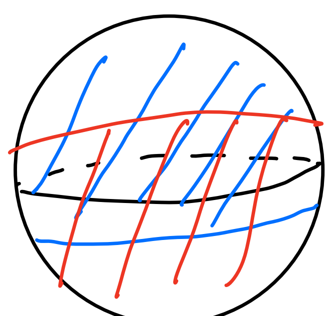
3) The sequence is exact.

<u>Homology groups of S^n</u>

$H_0(S^n) \cong \mathbb{Z}$

$H_1(S^n) = 0$, $n > 1$

as $\pi_1(S^n) = 0$



$H_0(S^1) \cong H_1(S^1) \cong \mathbb{Z}$.

Consider the pair $(X, A) = (\mathbb{D}^k, S^{k-1})$

$$H_n(\mathbb{D}^k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0 & n \geq 1 \end{cases}$$

$\mathbb{D}^k$ is contractible.

The corresponding long exact sequence from the thm is

$\because$ $H_n(\mathbb{D}^k) = \{0\} \implies$ every 3rd term in the l.e.s of $(\mathbb{D}^k, S^{k-1}; \mathbb{Z})$ is 0.

$$0 \longrightarrow H_{n+1}(\mathbb{D}^k, S^{k-1}; \mathbb{Z}) \xrightarrow{\partial_*} H_n(S^{k-1}; \mathbb{Z}) \longrightarrow 0$$

$\forall\, n \geq 1.$

will be exact $\iff$

∂_* is an iso. $\implies$

$$H_{n+1}(\mathbb{D}^k, S^{k-1}; \mathbb{Z}) \cong H_n(S^{k-1}; \mathbb{Z})$$

$\forall\, n \geq 1$

$\downarrow \cong$ Black box

$$\boxed{H_{n+1}(S^k; \mathbb{Z}) \cong H_n(S^{k-1}; \mathbb{Z})}$$

5

$$H_{n+1}(S^1;\mathbb{Z}) \cong H_n(\{pt\};\mathbb{Z}) = 0 \quad \forall\ n \geq 1$$

$$\Rightarrow H_m(S^1;\mathbb{Z}) \cong 0 \quad \forall\ n \geq 2$$

$$H_n(S^1) \cong \begin{cases} \mathbb{Z}, & n=0,1 \\ 0, & \text{otherwise} \end{cases}$$

From 3

$$H_n(S^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0,2 \\ 0, & \text{otherwise} \end{cases}$$

By induction

$$H_n(S^m;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0,m \\ 0, & \text{otherwise} \end{cases}$$

Exer. 1 Prove that $\mathbb{R}^n \cong \mathbb{R}^m \iff n=m$.

2 Brouwer's fixed thm for n-dim.
$f: D^n \to D^n$ continuous then f must have a fixed point.

Dr. Marc Kegel — Instructor.

11th Aug. 9-12

27th Sep. 9-12