Lecture 27

Recall :- (C_*, ∂) chain complex.

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$\partial^2 = 0$ or $\partial_{n-1} \circ \partial_n$ is the trivial hom.

$$H_n(C_*, \partial) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

$$H_*(C_*, \partial) = \bigoplus H_n(C_*, \partial)$$

Chain map (A_*, ∂_A) (B_*, ∂_B)

$f : A_* \to B_*$ chain map

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowleft"] & A_{n-1} \arrow[r, "\partial^A_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r, "\partial^B_{n-1}"] & \cdots
\end{tikzcd}$$

$$\partial^B \circ f = f \circ \partial^A.$$

If $f : (A_*, \partial^A) \to (B_*, \partial^B)$ is a chain map $\Rightarrow$

$$f_* : H_n(A_*, \partial^A) \to H_n(B_*, \partial^B) \quad \forall n$$

$$f_*([a]) = [f(a)].$$

Chain Homotopy

$$f, g : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$$

f is chain hom. to g if $\exists$ a sequence of hom.

$h_n : A_n \longrightarrow B_{n+1}$ s.t

$$f_n - g_n = \partial^B_{n+1} \circ h_n + h_{n-1} \circ \partial^A_n$$

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & A_n \arrow[r, "\partial"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & A_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] & B_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$\Big\}$

$$f_* = g_* : H_n(A_*, \partial^A) \longrightarrow H_n(B_*, \partial^B)$$

$H_n(X ; G)$ $\qquad$ $G = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_2, \mathbb{R}$ ..

↳ coefficient gp, abelian gp.

Δ^n $\quad$, $\quad \partial_{(k)} \Delta^n \cong \Delta^{n-1}$

X top. space. A singular n-simplex is a continuous map $\sigma : \Delta^n \longrightarrow X$.

$\mathcal{K}_n(X)$ the set of all singular n-simplices in X.

singular n-chain group

$$C_n(X ; G) = \bigoplus_{\sigma \in \mathcal{K}_n(X)} G$$

$\}$

$\sum a_i \sigma_i, \quad a_i \in G, \sigma_i \in K_n(X)$

$$\partial\sigma = \sum_{k=0}^{n} (-1)^k \left(\sigma|_{\partial(k)\Delta^n}\right) \in C_{n-1}(X;G)$$

$\}$ $\partial^2 = 0$

n-th singular hom. gp

$$H_n(X;G) = H_n(C_*(X;G), \partial)$$

<u>Lemma</u> :- Let X be a top space, G coefficient group.

$$H_0(X;G) \cong \bigoplus_{\substack{\#\text{ of path}\\ \text{components}\\ \text{of } X.}} G$$

<u>Proof</u> :- $\{\sigma : \Delta^0 \to X\} = K_0(X)$

$K_0(X) \cong X$

$\therefore$ the 0-chain can be written as $\sum a_i x_i$ w/ $a_i \in G$

$x_i \in X$.

$$\{\sigma : \Delta^1 \to X\} = \{\sigma : \underset{\substack{\shortparallel \\ [0,1]}}{I} \to X\}$$

any $\sigma \in K_1(X)$ can be viewed as a path $\sigma : I \to X$
and $\partial(\sigma) = \sigma(1) - \sigma(0)$.

every 0-chain is actually a cycle. ax and ay are 0-cycle then $(a \in G, \; x, y \in X)$

$$[ax] = [ay] \implies ax - ay = \partial(\sigma)$$

path

$$[ax] = [ay] \impliedby ax - ay = \partial(a\sigma)$$

is a path in X b/w x and y and $\implies$ x any y lie in the same path-component.

$\therefore$ if we pick up $x_\alpha \in X_\alpha$ path-component of X

then any 0-cycle is homologous to $\sum a_\alpha x_\alpha$, $a_\alpha \in G$

and $\therefore$ $H_0(X; G) \cong \bigoplus G$

# of path-components of X. $\boxtimes$

$\{$paths in X, $\sigma : I \to X\} = C_1(X)$

$$\partial\sigma = \sigma(1) - \sigma(0)$$

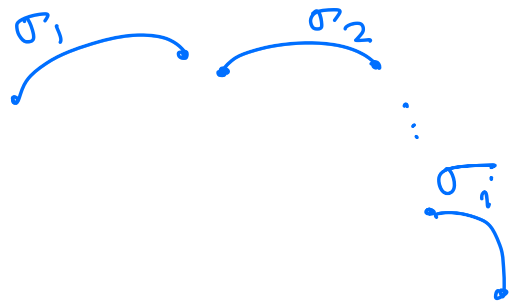
$\therefore$ if σ is a loop in X then $\partial\sigma = 0$, i.e., σ is a 1-cycle.

$G = \mathbb{Z}$. $\quad C_1(X; \mathbb{Z}) \ni \sum m_i \sigma_i \quad m_i \in \mathbb{Z}$, σ_i paths in X.

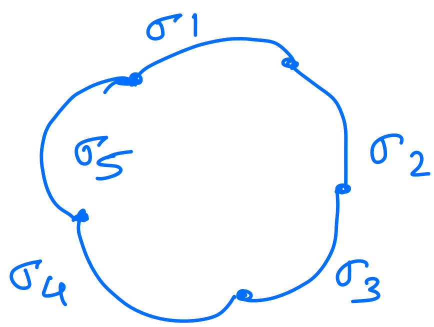
WLOG, assume $m_i = \pm 1$

We'll write $-\sigma_i$ for the reverse path σ_i^{-1}

$$\partial(-\sigma_i) = -(\sigma_i(1) - \sigma_i(0)) = \sigma_i(0) - \sigma_i(1)$$
$$= \partial(\sigma_i^{-1})$$



$\sum m_i \sigma_i$ will be a 1-cycle if we can concatenate the path σ_i together in such a way that each σ_i is concatenated w/ σ_{i+1} and the last path can be concatenated w/ the first.



Theorem:- Let X be a path-connected space w/ $x_0 \in X$. Then the bijection b/w singular 1-chain in X and path in X determines a group hom.

$h : \pi_1(X, x_0) \longrightarrow H_1(X;\mathbb{Z})$ w/ kernel is $[\pi_1(X,x_0), \pi_1(X,x_0)]$. Thus,

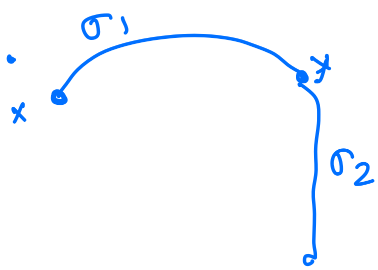
Hurewicz map

$$H_1(X;\mathbb{Z}) \cong \frac{\pi_1(X,x_0)}{[\pi_1(X,x_0),\pi_1(X,x_0)]}$$

abelianization of the fundamental group.

Heuristic Arguments:

$\sigma : I \to X \rightsquigarrow$ singular 1-chain $\tilde{h}(\sigma)$

- If σ is a loop then $\partial \tilde{h}(\sigma) = 0$.
- 

concatenated path $\sigma_1 * \sigma_2$

$$\tilde{h}(\sigma_1) + \tilde{h}(\sigma_2) - \tilde{h}(\sigma_1 * \sigma_2)$$
$$= \partial \tau$$, τ is a singular 2-simplex.

If σ_1 and σ_2 are homotopic to each other w/ fixed end points then $\tilde{h}(\sigma_1) - \tilde{h}(\sigma_2) = \partial \tau$, τ is a singular 2-chain.

$$h : \pi_1(X) \longrightarrow H_1(X;\mathbb{Z})$$
$$[\sigma] \longmapsto [\tilde{h}(\sigma)]$$

A cycle c is said to nullhomologous if $[c]=0$ i.e, $c = \partial \tau$.

$[\pi_1(X), \pi_1(X)] \subset \ker h$. Indeed $\ker h = [\pi_1(X), \pi_1(X)]$.

$\boxtimes$

Relative Homology

Prop.:- If $f: X \to Y$ is a continuous map then it induces a chain map $f_*: C_*(X;G) \to C_*(Y;G)$ as

$$f_*(\sigma) = f \circ \sigma \quad \text{∀ singular } n\text{-simplex } \sigma \text{ in } X.$$

$\sigma: \Delta^n \to X$

$$\partial \circ f_* = f_* \circ \partial$$

$g: Y \to Z$

$$(g \circ f)_* = g_* \circ f_*$$

composition of the chain maps

$id: X \to X \rightsquigarrow (id_*)$ at the chain level.

$\Rightarrow$ id. hom. at the hom. group level.

Theorem. [Homology groups are topological invariants]

If X and Y are homeomorphic then all of their homology groups are isomorphic.

Defn: A pair will be a tuple (X, A) where X

is a top.space and $A \subset X$.

(X, A) and (Y, B) be two pairs. A map $f: X \to Y$ is called a map of pairs if $f(A) \subset B$.

$$f : (X, A) \longrightarrow (Y, B).$$

$f, g : (X, A) \longrightarrow (Y, B)$ are homotopic if $\exists$ a homotopy $H : I \times X \to Y$ b/w f and g s.t $H(s, \cdot) : (X, A) \to (Y, B)$ is a map of pair $\forall s$. , i.e. $\underset{\forall s}{H(s, A)} \subset B$ or $H(I \times A) \subset B$.

$$f : (X, A) \longrightarrow (Y, B)$$
$$g : (Y, B) \longrightarrow (X, A)$$

If $g \circ f$ is homotopic as a map of pair to id : $(X, A) \to (X, A)$ and $f \circ g \simeq$ to id : $(Y, B) \to (Y, B)$ then we say that f and g are homotopy equivalence of pairs.

$$(X, \phi) \rightsquigarrow X$$

* $(X, A) \rightsquigarrow$ relative homology of the pair.

any singular n-simplex in $A \longrightarrow$ also a singular n-simplex in X.

$$\Downarrow$$

$$C_n(A;G) \leq C_n(X;G) \quad \forall n.$$

$$\begin{array}{c} \partial : C_n(X;G) \longrightarrow C_{n-1}(X,G) \\ \downarrow \\ C_n(A;G) \longrightarrow C_{n-1}(A,G) \end{array}$$

we have a well-defined boundary homomorphism ∂ on the quotient

$$C_n(X,A;G) = C_n(X;G) \Big/ C_n(A;G)$$

$$\partial^2 = 0$$

$\therefore$ $(C_*(X,A;G), \partial)$ is a chain complex known as relative singular chain complex of the pair (X,A).

The homology groups of $(C_*(X,A;G),\partial)$ are called relative singular homology groups.

$$H_n(X,A;G) = H_n(C_*(X,A;G),\partial)$$

Rem. When $A = \phi$ then we get back the sing. hom. gp of X called the absolute hom. groups of X.

- compute $H_n(S^n)$.