Lecture 26

Singular Homology

Defn and concepts being discussed are part of Homological algebra.

Defn

A ($\mathbb{Z}$-graded) chain complex of abelian groups (C_*, ∂) consists of a sequence of abelian groups $(C_n)_{n\in\mathbb{Z}}$ together w/ homomorphisms $\partial_n : C_n \longrightarrow C_{n-1}$ $\forall n \in \mathbb{Z}$ s.t. $\partial_{n-1} \circ \partial_n : C_n \longrightarrow C_{n-2}$ is the trivial hom. $\forall n$.

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$C_* = \bigoplus_{n\in\mathbb{Z}} C_n$$

$$\Sigma' a_i \, , \quad a_i \in C_{n_i} \, , \; n_i \in \mathbb{Z}$$

$$\partial : C_* \longrightarrow C_{*-1} \quad \text{degree } -1.$$

$\partial_{n-1} \circ \partial_n$ is the trivial hom $\Rightarrow$ $\partial^2 = 0$

$(\partial_n \circ \partial_{n+1} = 0)$

$$\operatorname{im} \partial_{n+1} \subset \operatorname{Ker} \partial_n \quad \forall n.$$

∂ – boundary operator, elements of $\operatorname{Ker} \partial \to$ cycles elements of $\ker \partial_n \rightsquigarrow$ n-cycles

elements of $\operatorname{im} \partial \rightsquigarrow$ boundaries.

Def^n:- The homology of a chain complex (C_*, ∂) is the sequence of abelian groups

$$H_n(C_*, \partial) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

$\forall\ n \in \mathbb{Z}$.

$$\underbrace{H_*(C_*, \partial)}_{\mathbb{Z}\text{-graded abelian group.}} = \bigoplus_{n \in \mathbb{Z}} H_n(C_*, \partial)$$

elements in $H_n(C_*, \partial)$ look like $[c]$ c is some n-cycle, $c \in \ker \partial_n$

$[c] \downarrow$ homology class of c.

$a, b \in \ker \partial_n$ are called homologous if $a - b = \partial_{n+1}(d)$ for some $d \in C_{n+1}$.

Def^n: Let (A_*, ∂^A) and (B_*, ∂^B) be two chain complexes. $f: A_* \longrightarrow B_*$ is called a chain map if it gives rise to a sequence of homomorphisms $f_n: A_n \longrightarrow B_n$ $\forall\ n \in \mathbb{Z}$ s.t. the following diagram commutes:-

$$
\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowright"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowleft"] & A_{n-1} \arrow[r, "\partial^A_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r, "\partial^B_{n-1}"] & \cdots
\end{tikzcd}
$$

$$f_n \circ \partial^A_{n+1} = \partial^B_{n+1} \circ f_{n+1} \quad \forall\ n \in \mathbb{Z}$$

$$\Downarrow$$

$$f \circ \partial^A = \partial^B \circ f \quad \text{------} \quad (1)$$

Prop:- Any chain map $f : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ induces homomorphism $f_* : H_n(A_*, \partial^A) \longrightarrow H_n(B_*, \partial^B)$ $\forall\ n \in \mathbb{Z}$

$$f_*([a]) = [f(a)]. \quad \text{------} \quad (2)$$

Proof:- Want:- If $a \in A_n$ is an n-cycle, i.e, $a \in \ker \partial^A_n$ then $f(a) \in B_n$ is again an n-cycle.

$$\partial^A_n(a) = 0$$

$$\Rightarrow \quad f \circ \partial^A_n(a) = 0 = \partial^B \circ (f(a)) = 0$$

$$\Rightarrow \quad f(a) \in \ker \partial^B \quad \Rightarrow \quad f(a) \text{ is also an } n\text{-cycle.}$$

To check that f_* is indeed a well-defined map, we check that f maps boundaries to boundaries.

$$\text{If} \quad a = \partial^A c$$

then $\because$ $f \circ \partial^A = \partial^B \circ f$

$\Rightarrow$ $f \circ \partial^A c = \partial^B \circ f(c)$ $\Rightarrow$ $f(c)$ is indeed a boundary.

$\therefore$ f_* given by (2) is well-defined. $\boxed{///}$

Defn:- A chain homotopy h b/w two chain maps $f, g : (A_*, \partial^A) \longrightarrow (B_*, \partial^A)$ is a sequence of homomorphisms $h_n : A_n \longrightarrow B_{n+1}$ s.t. $\forall\, n \in \mathbb{Z}$

$$f_n - g_n = \partial^B_{n+1} \circ h_n + h_{n-1} \circ \partial^A_n .$$

$$h : A_* \longrightarrow B_* \quad \text{s.t.} \quad f - g = \partial^B \circ h + h \circ \partial^A . \quad (3)$$

$h : A_* \longrightarrow B_{*+1}$

degree $+1$

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "f_{n+1}", "g_{n+1}"'] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "f_n", "g_n"'] \arrow[dl, "h_n"'] & A_{n-1} \arrow[r, "\partial^A_{n-1}"] \arrow[d, "f_{n-1}", "g_{n-1}"'] \arrow[dl, "h_{n-1}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r, "\partial^B_{n-1}"] & \cdots
\end{tikzcd}$$

In this case, f and g are called chain homotopic.

Prop:- Let $h : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain hom. b/w chain maps f and g. Then

$$f_* = g_* : H_n(A_*, \partial^A) \to H_n(B_*, \partial^A).$$

Proof Suppose $[a] \in H_n(A_*, \partial^A) \Rightarrow \partial^A a = 0$

$$\Rightarrow f(a) - g(a) = \partial^B h(a) + h \circ \partial^A(a)$$

$$= \partial^B(h(a))$$

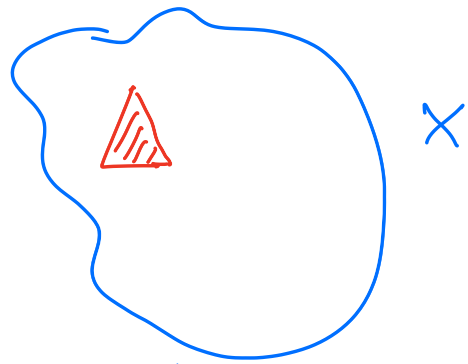
$\therefore$ $f(a) - g(a)$ is a boundary of the chain $h(a)$ in B_*

$$\Rightarrow [f(a)] = [g(a)]$$ $\boxtimes$

p-chain in simplicial hom. $c : \sigma \to \mathbb{Z}$ (G)

We can work w/ any abelian group G instead of $\mathbb{Z}$, called the coefficient group. In practice $G = \mathbb{Z}, \mathbb{Z}_2$ $(\mathbb{Z}_p)$, $\mathbb{Q}, \mathbb{R}$..

σ: n-simplex $\longrightarrow X$
↳ continuous.



The standard n-simplex is

$$\Delta^n = \Big\{ (t_0, t_1, \ldots, t_n) \in \underbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}_{n+1 \text{ times}} \;\Big|\; \sum_{i=0}^{n} t_i = 1 \Big\}$$

For $k = 0, \ldots, n$, the k-th boundary face of Δ^n is the subset

$$\partial_{(k)}\Delta^n = \{ t_k = 0 \} \subset \Delta^n$$
$$\cong$$
$$\Delta^{n-1}$$

Defn Let X be a top. space. A singular n-simplex in X is a continuous map $\sigma : \Delta^n \longrightarrow X$.

$\mathcal{K}_n(X)$ = set of all singular n-simplices in X

$= \{ \sigma : \Delta^n \longrightarrow X \mid \sigma \text{ is cont.} \}$

G is an abelian group.

group of singular n-chain $C_n(X; G) = \bigoplus_{\sigma \in \mathcal{K}_n(X)} G$

elements in $C_n(X;G)$ as finite sums $\sum_i a_i \sigma_i$
w/ $a_i \in G$, $\sigma_i \in \mathcal{K}_n(X)$.

We also have boundary maps $\partial : C_n(X;G) \longrightarrow C_{n-1}(X;G)$

$$\partial \sigma := \sum_{k=0}^{n} (-1)^k \left(\sigma|_{\partial_{(k)}\Delta^n} \right) \in C_{n-1}(X;G)$$

cont. map from
$\Delta^{n-1} \longrightarrow X$

$$\partial_n : C_n(X;G) \longrightarrow C_{n-1}(X;G)$$

$$\sum a_i \sigma_i \overset{\partial_n}{\rightsquigarrow} \sum a_i \partial \sigma_i$$

$\boxed{\partial^2 = 0}$ holds here as well.

Def^n:- The n-th singular homology group w/ coefficients in G is

$$H_n(X;G) = H_n\left(C_*(X;G), \partial \right)$$

When $G = \mathbb{Z}$, $H_n(X)$.

* Rel. b/w $H_1(X, \mathbb{Z})$ and $\pi_1(X)$ $\left\{ \begin{array}{l} H_n(X, \mathbb{Z}) \sim \\ \pi_n(X) \end{array} \right\}$
* How do we compute $H_n(X)$?

  Relative homology, Excision.