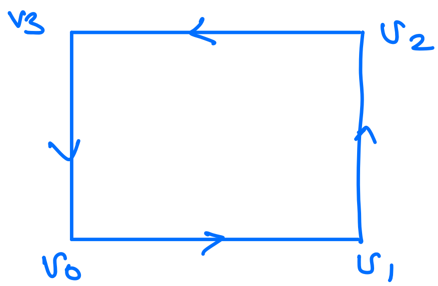
Lecture 25 (during the problem session)

Examples :-



Y

$H_0(Y) \cong \mathbb{Z}$

$H_n(Y) = 0 \quad \forall\, n \geq 2.$

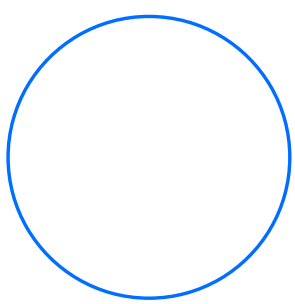
$C_1(K)$ is a free abelian group of rank 4

$$\partial_1\Big(n_1[v_1, v_0] + n_2[v_2, v_1] + n_3[v_3, v_2] + n_4([v_0, v_3])\Big) = 0$$

$$n_1 = n_2 = n_3 = n_4$$

$$H^1(Y) \cong \mathbb{Z}.$$

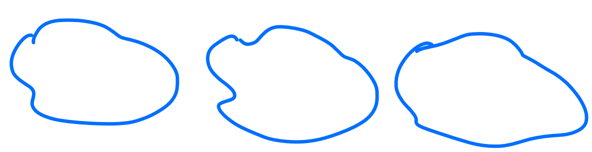
- n-th homology group $H_n(X)$ detects the "n-dimensional" holes in X.

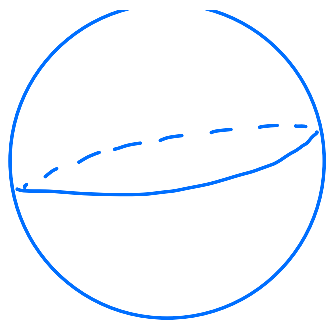


S^1

$H_0(X) \rightsquigarrow$ connected components of X.

$H_0(X) \cong \mathbb{Z}^d$, d is the # of connected components.



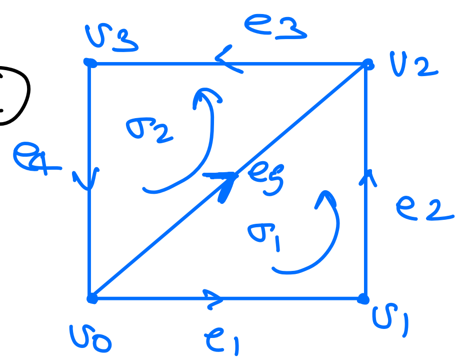


$H_0(S^2) \simeq \mathbb{Z}$
$H_1(S^2) = \{0\}$

$H_2(S^2) \cong \mathbb{Z}$

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i = 1, \dots, n-1 \\ \mathbb{Z}, & i = n \\ 0, & i > n \end{cases}$$

2



L

$H_0(L) \cong \mathbb{Z}$

a 1-chain is of the form

$\sum n_i e_i$

$$-\partial_1\big(n_1[v_1, v_0] + n_2[v_2, v_1] + n_3[v_3, v_2] + n_4[v_0, v_3] + n_5[v_2, v_0]\big)$$

$$= -[n_1(v_1 - v_0) + n_2(v_2 - v_1) + n_3(v_3 - v_2) + n_4(v_0 - v_3) + n_5(v_2 - v_0)]$$

$$= -[v_0(-n_1 + n_4 - n_5) + v_1(n_1 - n_2) + v_2(n_2 - n_3 + n_5) + v_3(n_3 - n_4)]$$

$Z_1(L) = \ker(\partial_1)$

$-n_1+n_4-n_5=0 \Rightarrow n_5 = -n_1+n_4$

$n_1=n_2$

$n_2-n_3+n_5=0 \Rightarrow n_5 = n_3-n_2$

$n_3=n_4$ $\quad (n_1,n_2,n_3,n_4,n_5) \rightsquigarrow \begin{array}{l}(1,1,0,0,-1)\\(0,0,1,1,1)\end{array}$

$\therefore$ The degrees of freedom are 2

$\Rightarrow$ $Z_1(L)$ is a free abelian group of rank 2. An explicit basis is

$$\underset{\partial_2(\sigma_1)}{\underset{\shortparallel}{e_1+e_2-e_5}} \quad \text{and} \quad \underset{\partial_2(\sigma_2)}{\underset{\shortparallel}{e_3+e_4+e_5}}.$$

$B_1(L) = \operatorname{Im} \partial_2 : C_2(L) \to C_1(L)$

$H_1(L) = \dfrac{Z_1(L)}{B_1(L)} = 0$ b/c the basis elements of $Z_1(L)$ are actually boundaries of 2-chains.

$H_0(L) \cong \mathbb{Z}$

$H_i(L) = 0 \quad , i \geq 1.$

L is a simplicial complex representation of $\mathbb{B}^2$.

$\partial_2(m_1\sigma_1 + m_2\sigma_2)$, $m_1, m_2 \in \mathbb{Z}$

$\shortparallel$

$0 \iff m_1 = m_2 = 0.$

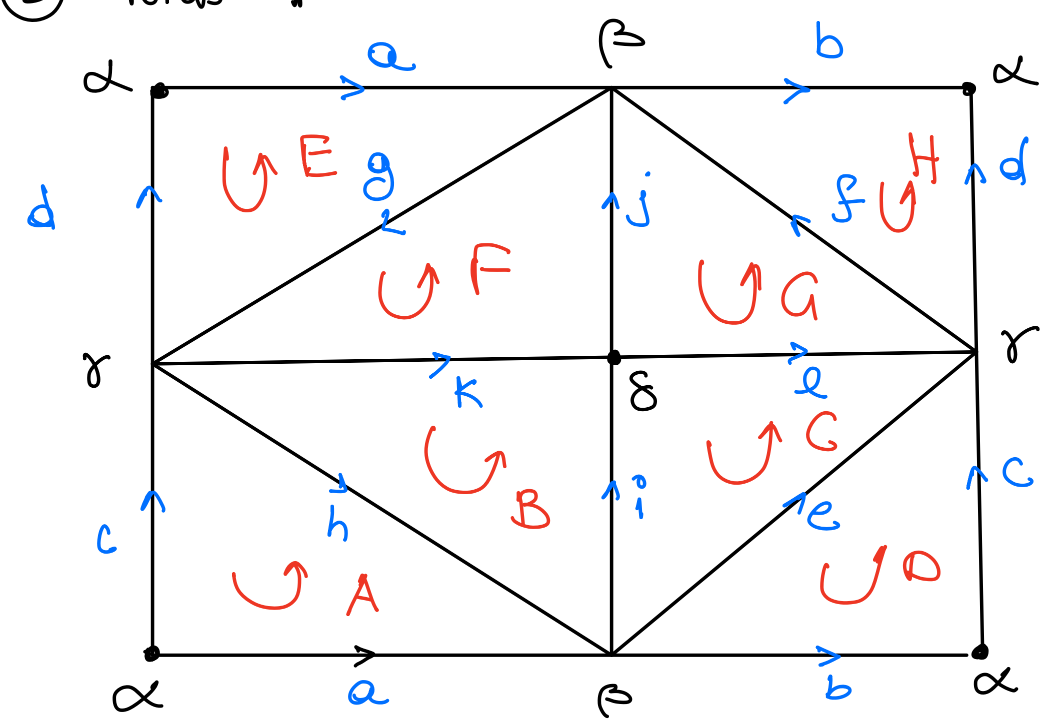
- As $\dim(K) \uparrow$ the calculation of $H_n(K)$ becomes more and more tedious.

<u>Cellular Homology</u> $\rightsquigarrow$ more tractable

We say that two p-chains c and c' are homologous if $c - c' = \partial_{p+1} d$ for some $p+1$-chain d.

If $c = \partial_{p+1} d$ we say that c is homologous of 0.

③ Torus $\mathbb{T}^2$



$\partial A = a - h - c$

$\partial B = i - k + h$

$\partial(n_1 A + n_2 B) = 0$

T

$[V + F - E = \chi(\mathbb{T}^2)]$

C_0 = rank 4
C_1 = rank 12
C_2 = rank 8

e.g. $\partial_2 A = a - h - G$
$\partial_2 G = l + f - j$
$\partial_1 a = \beta - \alpha$

$Z_0(T) \cong C_0$

all the generators of C_0 (Z_0) are homologous to each other, i.e, diff. of any 2 of them is the boundary of 1-simplex.

$[\alpha]$ is non-zero ie Z_0/B_0

$\therefore$ $H_0(T^2) \cong \mathbb{Z}$ (expected as T^2 is connected)

e.g. the square in the lower-right hand is actually a 1-cycle b/c

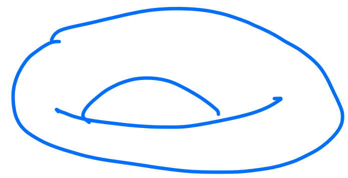
$\partial_1(i + l - c - b) = 0$ is also boundary of a 2-simple $(C+D)$

$\partial_1(i+j) = \partial_1(k+l) = 0$

when you look at the quotient of $Z_1(T)/B_1(T)$, the elements $[i+l-c-b]$ will be 0.

[i+j] and [k+l] are non-zero elements.

$\therefore \quad H_1(T) \cong \mathbb{Z} \times \mathbb{Z}$



for calculating :-

$$H_2(T) = Z_2(T) \big/ B_2(T)$$

$$\partial(\underline{A+B+C+D+E+F+G+H}) = 0$$

$$Z_2(T) \leq C_2(T)$$

$$H_2(T^2) = Z_2(T) \big/ \{0\} \cong \mathbb{Z}$$

$$H_i(\mathbb{T}^2) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z} \times \mathbb{Z}, & i = 1 \\ \mathbb{Z}, & i = 2 \\ 0, & i \geq 3. \end{cases}$$

Singular Homology

Def^n:- A ($\mathbb{Z}$-graded) chain complex of abelian groups (C_*, ∂) consists of sequence $\{C_n\}$ of abelian together w/ homomorphism $\partial_n : C_n \to C_{n-1}$ $\forall$ $n \in \mathbb{Z}$ s.t. $\partial_{n-1} \circ \partial_n : C_n \to C_{n-2}$ is the trivial homomorphism $\forall$ n.

$$C_* = \bigoplus_{n \in \mathbb{Z}} C_n$$

$$C_* = \Big\{ \sum a_i \;\Big|\; a_i \in C_{n_i}, \; n_i \in \mathbb{Z} \Big\}$$

all but finitely many terms are zero.

$x \in C_*$, $\deg(x) = n$ if $x \in C_n$.

$\partial_n : C_n \to C_{n-1}$ $\rightsquigarrow$ $\partial : C_* \to C_*$ has degree -1.

$$\partial : C_* \longrightarrow C_{*-1}$$

$$\underbrace{\partial^2 = 0}_{} \rightsquigarrow \partial_{n-1} \circ \partial_n = 0 \quad \forall n \in \mathbb{Z}$$

$\Downarrow$

$\mathrm{im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$ ∂ - boundary operators

elements of $\ker(\partial)$ are called cycles.

" ——" im(∂) " —" boundaries.