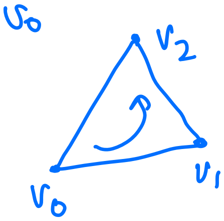
Lecture 24

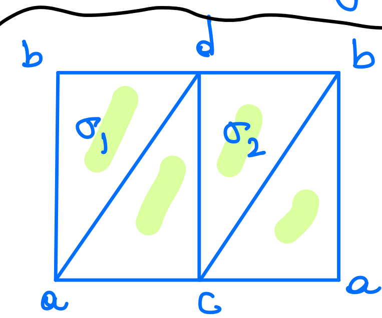
- Pset 8 has been posted.

Recall:-

two orderings of the vertex set of a simplex σ are equivalent if they differ by an even permutation.

$[v_0, v_1, \dots, v_n] \rightsquigarrow$ oriented simplex spanned by $v_0, \dots, v_n$.





$\sigma_1 \cap \sigma_2 = \{b, d\}$ contradicts the def^n of a simplicial complex.

Def^n:- Let K be a simplicial complex. A p-chain on K is a function c from the set of oriented p-simplices in K to $\mathbb{Z}$ s.t.

① $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.

② $c(\sigma) = 0$ for all but finitely many p-simplices σ.

We can add p-chains by adding their values:

$C_p(K) \rightsquigarrow$ group of p-chains of K.

If $p<0$ or $p>\dim K$ then $C_p(K) = \{e\}$.

If σ is an oriented simplex, the elementary chain c correspon-
-ding to σ is the function:-

$c(\sigma) = 1$

$c(\sigma') = -1$ if σ' has opposite orientation than σ

$c(\tau) = 0$ for all other simplex τ.

<u>Lemma</u>:- $C_p(K)$ is a free abelian group; a basis for $C_p(K)$ can be obtained by orienting each p-simplex and using the corresponding elementary chains as a basis.

<u>Proof</u> If $\{\sigma_i\}$ all oriented p-simplices of K then an arbitrary p-chain c can be written as

$$c = \sum n_i \sigma_i \quad , \quad n_i \in \mathbb{Z}$$

$$C_p(K) = \left\{ \sum n_i \sigma_i \;\middle|\; \begin{array}{l} n_i \in \mathbb{Z} \\ \sigma_i \text{ is an oriented } p\text{-simplex.} \\ \text{all but finitely many terms} \\ \text{are zero} \end{array} \right\}.$$

$C_0(K)$, $C_1(K)$, $C_2(K), \ldots,$

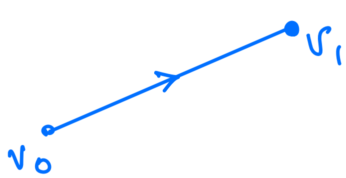
Cor. Any function f from the oriented p-simplices of K to an abelian group G extends uniquely to a homomorphism $C_p(K) \to G$, provided that
$f(-\sigma) = -f(\sigma)$ $\forall$ oriented p-simplex σ.
denotes the simplex σ w/ opposite orientation.

Defn:- We define a homomorphism
$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$
called the boundary operator. If $\sigma = [v_0, \ldots, v_p]$ is an oriented p-simplex w/ $p > 0$ then

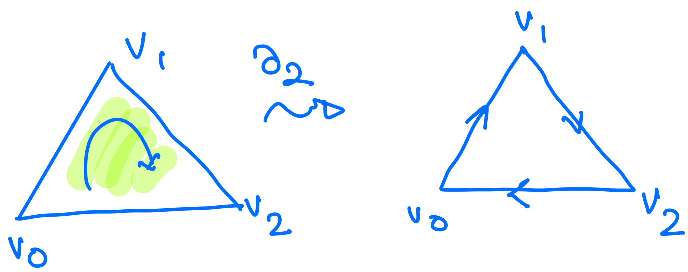
$$\partial_p \sigma = \partial_p [v_0, v_1, \ldots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_p] \tag{I}$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted.
$\because$ $C_p(K) = \{e\}$ for $p < 0$, ∂_p is the trivial homomorphism.

∂_1



$(-1)^0 [\hat{v}_0, v_1] + (-1)^1 [v_0, \hat{v}_1]$
$= v_1 - v_0$



$$\partial_2([v_0, v_1, v_2]) = (-1)^0 [\hat{v}_0, v_1, v_2] + (-1)^1 [v_0, \hat{v}_1, v_2] + (-1)^2 [v_0, v_1, \hat{v}_2]$$

$$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Want to check :- $\partial_p(-\sigma) = -\partial_p(\sigma)$.

$\underline{\partial_p[v_0, v_1, \dots, v_j, v_{j+1}, \dots, v_p]}$ and $\underline{\partial_p[v_0, \dots v_{j+1}, v_j, \dots, v_p]}$

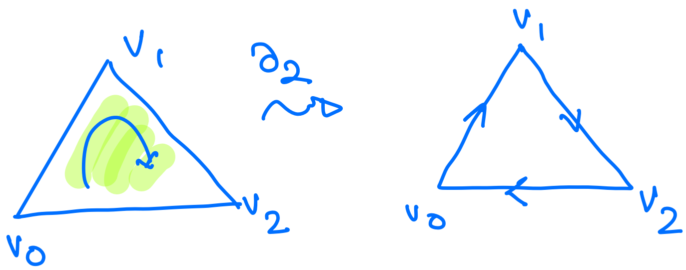
If $i \neq j, j+1$ $\rightsquigarrow$ the 2nd expression is $-$ of the 1st expressions.

if $i = j$ and $j+1$

$$(-1)^j [\dots v_{j-1}, \hat{v}_j, v_{j+1}, v_{j+2}, \dots] + (-1)^{j+1} [\dots, v_{j-1}, v_j, \hat{v}_{j+1}, \dots, v_p]$$

$$(-1)^j [\dots, v_{j-1}, \hat{v}_{j+1}, v_j, \dots, v_p] + (-1)^{j+1} [\dots, v_{j-1}, v_{j+1}, \hat{v}_j, \dots, v_p]$$

$\because$ the 2 expressions are differing by a sign $\Rightarrow$
$\partial_p(-\sigma) = -\partial_p(\sigma)$.



$$\partial_2([v_0, v_1, v_2]) = (-1)^0[\hat{v}_0, v_1, v_2] + (-1)^1[v_0, \hat{v}_1, v_2] + (-1)^2[v_0, v_1, \hat{v}_2]$$
$$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Let's calculate

$$\partial_1\Big[\partial_2([v_0, v_1, v_2])\Big] = \partial_1([v_1, v_2]) - \partial_1([v_0, v_2]) + \partial_1([v_0, v_1])$$
$$= v_2 - v_1 - (v_2 - v_0) + v_1 - v_0$$
$$= \cancel{v_2} - \cancel{v_1} - \cancel{v_2} + \cancel{v_0} + \cancel{v_1} - \cancel{v_0}$$
$$= 0$$

Lemma :- $\partial_{p-1} \circ \partial_p = 0$. ("Boundary of a boundary is 0").

Proof :-

$$\partial_{p-1}\left(\partial_p[v_0, v_1, \dots, v_p]\right)$$

$$= \partial_{p-1}\left(\sum_{i=0}^{p} (-1)^i [v_0, v_1, \dots, \hat{v}_i, v_{i+1}, \dots, v_p]\right)$$

$$= \sum_{i=0}^{p} (-1)^i \partial_{p-1}[v_0, v_1, \dots, \hat{v}_i, v_{i+1}, \dots, v_p]$$

$$= \sum_{j<i} (-1)^i (-1)^j [\dots, \hat{v}_j, \dots, \hat{v}_i, \dots]$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} [\dots, \hat{v}_i, \dots, \hat{v}_j, \dots]$$

The terms cancel each other and we get 0.

$\therefore$ $\partial_p \circ \partial_{p+1} = 0$. $\boxed{\checkmark}$

Defn:- $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$. The Kernel of ∂_p is called the group of p-cycles and is denoted by $Z_p(K) = \ker(\partial_p)$.

The image of $\partial_{p+1} : C_{p+1}(K) \longrightarrow C_p(K)$ is called the group of p-boundaries and is denoted by $B_p(K)$.

$$Z_p(K) \leq C_p(K)$$
$$B_p(K) \leq C_p(K)$$

$\because \quad \partial_p(\partial_{p+1}) = 0 \Rightarrow \partial_p(B_p(K)) = 0$

$\therefore \quad B_p(K) \leq Z_p(K)$

We define

$$H_p(K) = Z_p(K) / B_p(K)$$

called the p-th simplicial homology group of K.

$$H_0(K) = \frac{Z_0(K)}{B_0(K)}, \quad H_1(K) = \frac{Z_1(K)}{B_1(K)}, \ldots$$

Remark :-

$$\therefore \ldots C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \xrightarrow{\partial_{p-1}} C_{p-2}(X) \xrightarrow{\partial_{p-2}} C_{p-3}(X) \xrightarrow{\partial_{p-3}} \cdots$$

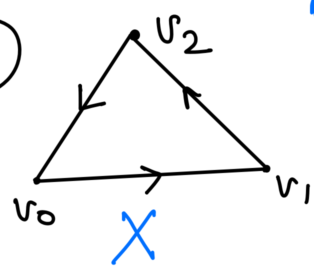
$$\partial \circ \partial = 0 \Leftrightarrow \partial^2 = 0 \qquad \text{"}\partial_p \circ \partial_{p+1}\text{"}$$

We can make sense of "homology groups".

- d - exterior derivative, $d \circ d = d^2 = 0$.

  $\rightsquigarrow$ de Rham cohomology groups.

Examples :-

1. 

Want to calculate $H_n(X)$.

$C_2(X) = \{0\}$

$H_2(X) = \{0\}$

$$H_n(X) = \{0\} \quad \forall \; n \geq 2.$$

$$H_1(X) = \frac{Z_1(X)}{B_1(X)} \qquad Z_1 = \text{Ker}: \partial_1 : C_1 \longrightarrow C_0$$

C_1 has basis $\{[v_0, v_1], [v_1, v_2], [v_2, v_0]\}$.

$$\partial_1([v_0, v_1]) = v_1 - v_0$$
$$\partial_1([v_1, v_2]) = v_2 - v_1$$
$$\partial_1([v_2, v_0]) = v_0 - v_2$$

for $n_1, n_2, n_3 \in \mathbb{Z}$

$$\partial_1(\underbrace{n_1[v_0,v_1] + n_2[v_1,v_2] + n_3[v_2,v_0]}_{\sigma})$$

$$= n_1(v_1 - v_0) + n_2(v_2 - v_1) + n_3(v_0 - v_2)$$

$$= v_0(n_3 - n_1) + v_1(n_1 - n_2) + v_2(n_2 - n_3)$$

for $\sigma \in \text{Ker}(\partial_1)$ $\partial = 0$

$$\begin{aligned} n_3 &= n_1 \\ n_1 &= n_2 \\ n_2 &= n_3 \end{aligned} \implies n_1 = n_2 = n_3$$

$$Z_1(X) = \text{Ker}(\partial_1) \cong \mathbb{Z} = \langle [v_0, v_1] + [v_1, v_2] + [v_2, v_0] \rangle$$

$$B_1(X) : \text{Im}\left(\partial_2 : \underset{\{0\}}{\underset{\parallel}{C_2(X)}} \longrightarrow C_1(X)\right)$$

$$\therefore B_1(X) = \{0\}$$

$H_1(X) \cong \mathbb{Z}.$

$H_0(X) = \dfrac{Z_0}{B_0}$ $\qquad Z_0 = \text{Ker}.\ \partial_0 : C_0 \longrightarrow \underbrace{C_{-1}}_{\{e\}}$

$\therefore\ Z_0(X) = C_0(X)$

(This is always the case for any simplicial complex)

$B_0 := \text{Im}(\partial_1 : C_1 \longrightarrow C_0)$

$\partial_1([v_0, v_1]) = v_1 - v_0$

$\partial_1([v_1, v_2]) = v_2 - v_1$

$\partial_1([v_2, v_0]) = v_0 - v_2$

$\text{Im}(\partial_1)$ is generated by

v_0, v_1, v_2 are equal mod $\text{Im}(\partial_1)$

$(a + H = b + H \iff a - b \in H)$

$n[v_0] = \partial_1(n_1[v_0, v_1] + n_2[v_1, v_2] + n_3[v_2, v_0])$

Only possible w/ $n = 0$.

$\therefore$ coset of $[v_0]$ generates the group Z_0/B_0

$\implies H_0(X) \cong \mathbb{Z}.$ $\qquad \boxtimes$

Remarks:- X is a simplicial complex repr. of the circle so we have calculated the homology groups of S^1.

$H_0(S^2) \cong H_2(S^2) \cong \mathbb{Z}$, $H_n(S^2) = 0 \quad \forall\, n \geq 3$

$n = 1$.