# Lecture 23

Recall :-

$\mathcal{S}$ - abstract simplicial complex

↳ collection of finite non empty sets s.t. if $A \in \mathcal{S}$

then $B \in \mathcal{S}$ for all $B \subseteq A$, $B \neq \phi$.

$A \in \mathcal{S} \to$ simplex

$\dim(A) = |A| - 1$ , any nonempty subset of $A$ is a face of $A$.

vertex set $V$ of $\mathcal{S}$ = union of the one-point elements of $\mathcal{S}$.

$\mathcal{S} \cong \mathcal{T}$ if $\exists$ bijective correspondence $f$ mapping the vertex set of $S$ to the vertex set of $\mathcal{T}$ s.t

$\{a_0, a_1, \ldots, a_n\} \in \mathcal{S} \iff \{f(a_0), f(a_1), \ldots, f(a_n)\} \in \mathcal{T}$.

Defn:- If $K$ is a simplicial complex, let $V$ be the vertex set of $K$. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \ldots, a_n\}$ of $V$ s.t the vertices $a_0, \ldots, a_n$ span a simplex of $K$.

$\mathcal{K}$ is called a vertex scheme of $K$.

Theorem :-

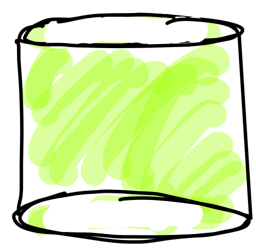
(a) Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex $K$.

(b) Two simplicial complexes are isomorphic $\Longleftrightarrow$ their vertex schemes are isomorphic as abstract simplicial complexes.

If the abstract complex $\mathcal{S} \cong$ vertex scheme of $K$ then we call $K$ a geometric realization of $\mathcal{S}$.

Examples :-

① cylinder $= S^1 \times I$



- it could happen that there are more than 1 simplicial complex representation of a given space.

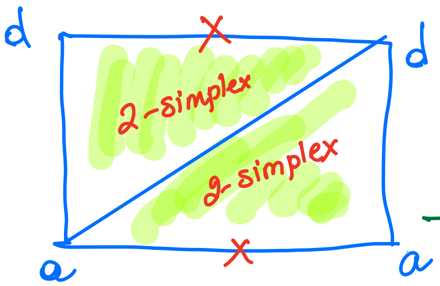
- intersection of any two simplices in $K$ must be either empty or a face of each.

- If a simplex has vertices $\{v_0, \dots, v_n\} \rightsquigarrow$ uniquely identify that $n$-simplex.

$\Updownarrow$ each simplex must be embedded, e.g. a 1-simplex cannot have the same end-point

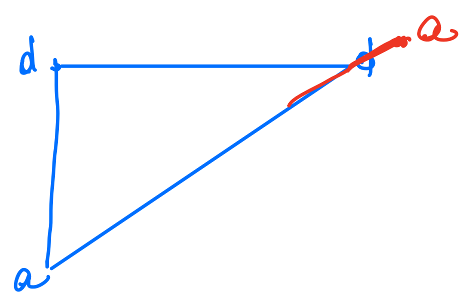

Whatever simplicial complex we draw for the cylinder, it must have a 2-simplex.

(1)

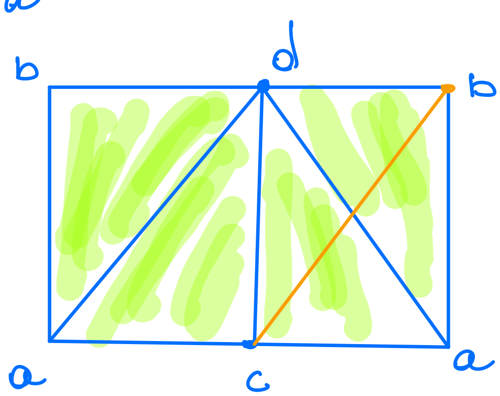



Is this a simplicial complex representation of a cylinder?

This is NOT a simplicial complex.




(2)

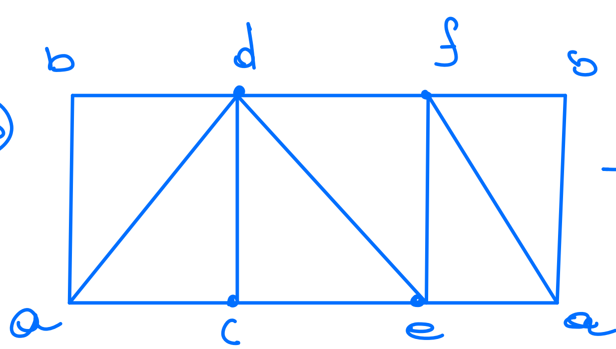



Is this a simplicial complex?

NO

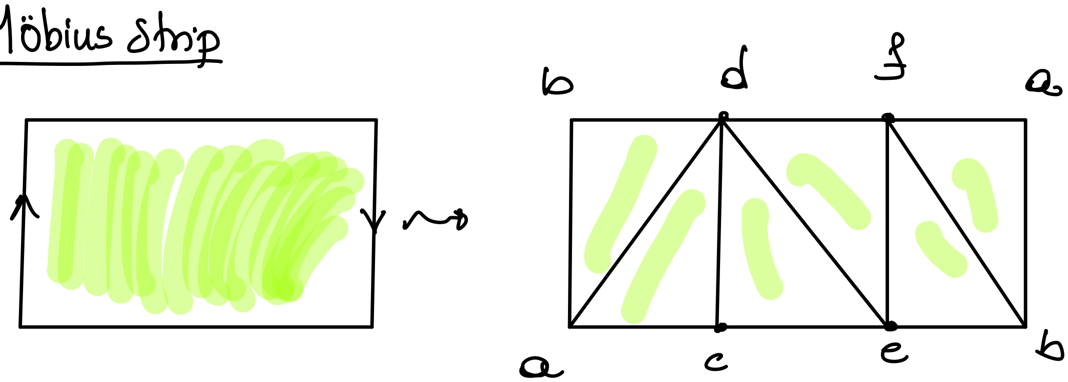
NO (exercise)

(3)



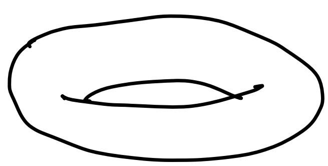

— This is a simplicial complex and is a simplicial complex representation of the cylinder.

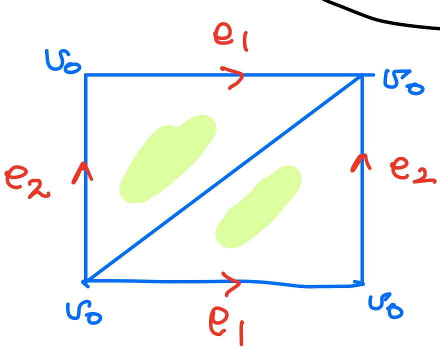
(2) **Möbius strip**



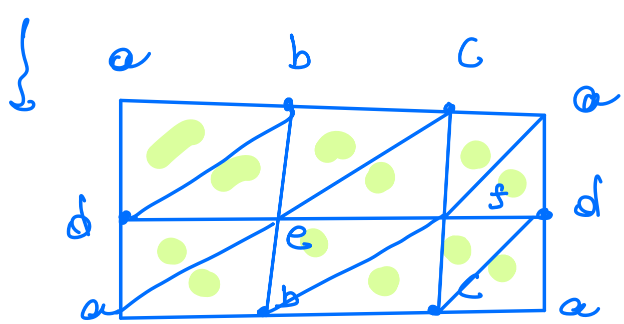

Simplicial complex rep. of a Möbius strip.

(3) **Torus**

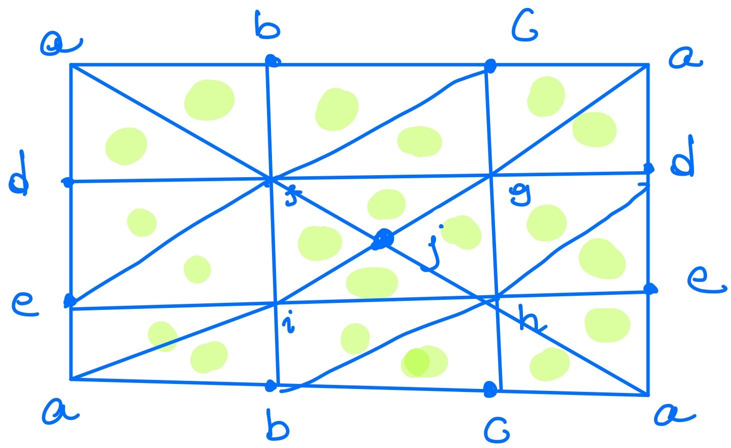





NOT a simplicial complex representation of T




NOT a s.c. r. of a torus.




• cellular homology.

Simplicial complex rep of a torus.

# Simplicial Homology Groups

**Def$^n$** Suppose $\sigma$ is a simplex. We define two orderings of its vertex set to be equivalent if they differ from each other by an even permutation.

$v_0\ v_1\ v_2\ v_3$

$v_1\ v_0\ v_2\ v_3$

$v_1\ v_2\ v_0\ v_3$

$v_1\ v_2\ v_3\ v_0$

odd permutation

$=$

$v_1\ v_2\ v_3\ v_0$

$v_0\ v_1\ v_2 \rightarrow v_1\ v_2\ v_0$

$v_1\ v_0\ v_2$

$v_1\ v_2\ v_0$

even permutation.

If $\dim \sigma > 0$, the orderings of the vertices of $\sigma$ fall into two equivalence classes.

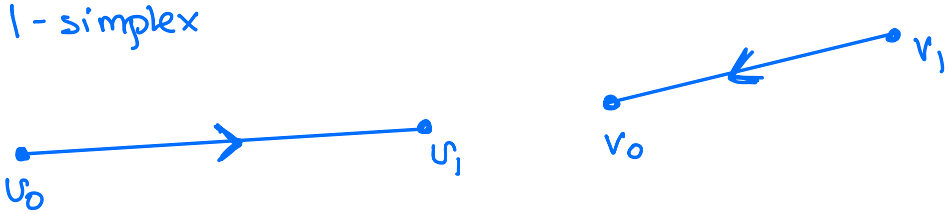
Each of these classes is called an orientation of $\sigma$.

An oriented simplex is a simplex $\sigma$ together w/ an orientation of $\sigma$.

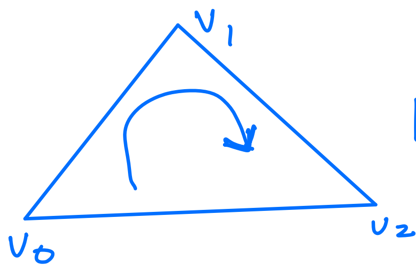
$v_0 \dots v_n$

$[v_0, v_1, \dots, v_n]$ – oriented simplex $\sigma$ spanned by $v_0, \dots, v_n$ and the given ordering.

1-simplex




2-simplex




$[v_0, v_1, v_2]$

$[v_1, v_2, v_0]$ $\quad$ $[v_2, v_0, v_1]$

$[v_1, v_0, v_2]$ – opposite orientation than the previous.

**Defⁿ** Let $K$ be a simplicial complex. A p-chain on $K$ is a function $c : \{\text{oriented } p\text{-simplices in } K\} \longrightarrow \mathbb{Z}$

s.t.

(1) $c(\sigma) = -c(\sigma')$ if $\sigma$ and $\sigma'$ are the same simplexes w/ opposite orientation.

(2) $c(\sigma) = 0$ for all but finitely many oriented $p$-simplices $\sigma$.

We add p-chain by adding their values.

The resulting group is called the group of oriented p-chains $C_p(K) = \left\{ \sum_{i=1}^{n} c_i \sigma_i \;\middle|\; c_i : \{\text{oriented } p\text{-simplices}\} \to \mathbb{Z} \right\}$

→ If $p < 0$ or $p > \dim K$ then $C_p(K)$ = trivial group.

→ If $\sigma$ is an oriented simplex, the elementary chain $c$ corresponding to $\sigma$ is the function, defined as:-

$c(\sigma) = 1$

$c(\sigma') = -1$ if $\sigma'$ and $\sigma$ have opposite orientation

$c(\tau) = 0$ for all other oriented $p$-simplex $\tau$.

→ $C_p(K)$ is a free abelian group; a basis for $C_p(K)$ can be obtained by orienting each $p$-simplex and use the corresponding elementary chains as a basis.