Lecture 22

- no prob. set this week. Next week's problem ~~session~~ will be like a lecture.

Recall :-

- K simplicial complex $\iff$ 1) every face of a simplex σ of K is in K.

  2) The intersection of any two simplexes of K must be a face of each of them

  $\Updownarrow$

  every pair of distinct simplices of K have disjoint interiors.

- p-skeleton of K $\sim K^{(p)}$ is the collection of all simplices of K of dim. at most p.

  $K^{(0)}$ = set of vertices of K.

- $|K| \subseteq \mathbb{R}^N$ which is the union of the simplices of K.

  $\overset{\cup\!\mid}{A}$ is closed in $|K| \iff A \cap \sigma$ is closed in σ $\forall$ $\sigma \in K$.

  polytope of K.

A space that is the polytope of a simplicial complex will be called a polyhedron.

Lemma If L is a subcomplex of K, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$ then σ is a closed subspace of $|K|$.

Proof:- Let B is closed in $|K| \Rightarrow B \cap \sigma$ is closed in σ $\forall\ \sigma \in K$ and $\therefore$ $\forall\ \sigma \in L$.

$\Downarrow$

$B \cap |L|$ is closed in $|L|$.

Conversely, A is closed in $|L|$. Let σ is a simplex of $K \Rightarrow$ $\sigma \cap |L|$ is the union of all the faces s_i of σ that belong to L.

$\because$ A is closed in $|L| \Rightarrow A \cap s_i$ is closed in s_i

$\Rightarrow$ $A \cap s_i$ is closed in σ.

$\Rightarrow$ $A \cap \sigma = \bigcup_{i=1}^{n} A \cap s_i$ is closed in σ $\forall\ \sigma \in K$

$\Rightarrow$ A is closed in $|K|$.

$\therefore$ $|L|$ is a closed subspace of $|K|$. $\blacksquare$

Lemma A map $f : |K| \longrightarrow X$ is continuous $\Leftrightarrow$ $f|_\sigma$ is continuous $\forall\ \sigma \in K$.

Proof. Let f is continuous $\Rightarrow f|_\sigma$ is continuous.

Conversely, suppose $f|_\sigma$ is cont. $\forall\ \sigma \in K$.

Let C is a closed set of $X \Rightarrow f^{-1}(C) \cap \sigma = \underbrace{(f|_\sigma)^{-1}(C)}_{\text{is closed}}$

$\Rightarrow$ $f^{-1}(C) \cap \sigma$ is closed in $|K|$ $\forall\ \sigma \in K$

$\Rightarrow$ $f : |K| \longrightarrow X$ is continuous. $\boxed{\checkmark}$

Defn If $x \in |K|$ then x is interior to precisely one simplex in K whose vertices $a_0, \ldots, a_n$. Then

$$x = \sum_{i=0}^{n} t_i a_i \qquad t_i > 0\ \forall\ i$$
$$\sum t_i = 1$$

If v is an arbitrary vertex of K, we define the barycentric coordinates $t_v(x)$ of x w.r.t. v by setting $t_v(x) = 0$ if $v \neq a_i$, $i = 0, \ldots, n$ and $t_v(x) = t_i$ if $v = a_i$

- v fixed, $t_v(x)$ is continuous func. restricted to a fixed simplex σ of K.

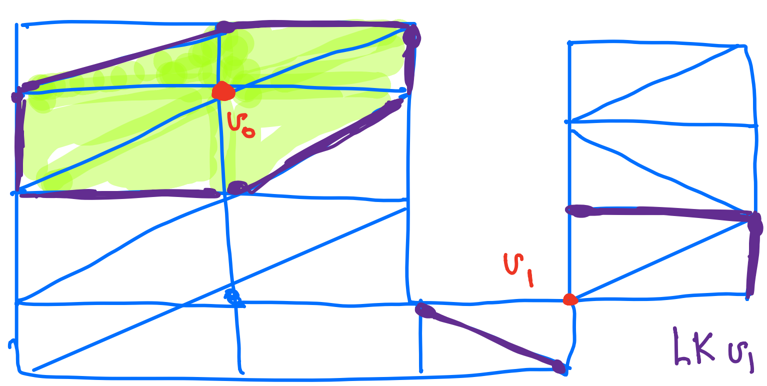
Important subspaces of $|K|$

Defn If v is a vertex of K, the star of v in K, denoted by $St\, v$ or $St(v, K)$ is the union of the interiors of those simplices of K that have v as a vertex.

The closure of $\mathrm{St}\, v$ is called the closed star of v in K. This is the union of all simplices of K which of v as a vertex and is the polytope of a subcomplex of K.

The set $\overline{\mathrm{St}\, v} - \mathrm{St}\, v$ is called the link of v in K and is denoted by $\mathrm{Lk}\, v$.

$v_0 \notin \mathrm{Lk}\, v_0$!



Simplicial maps

Lemma:- Let K and L be simplicial complexes and let $f : K^{(0)} \longrightarrow L^{(0)}$ be a map.

Suppose that whenever the vertices $v_0, \ldots, v_n$ of K span a simplex in K, the points $f(v_0), f(v_1), \ldots, f(v_n)$

are vertices of a simplex of L. Then f can be extended to a continuous map $g : |K| \longrightarrow |L|$ s.t.

$$x = \sum_{i=0}^{n} t_i v_i \implies g(x) = \sum_{i=0}^{n} t_i f(v_i) .$$

g is called the linear simplicial map induced by the vertex map f.

<u>Proof</u>:- Exercise.

☒

- composition of simplicial maps is a simplicial map.

<u>Lemma</u>:- Suppose $f : K^{(0)} \longrightarrow L^{(0)}$ is a bijective map s.t. the vertices $v_0, \dots, v_n$ of K span a simplex of $K \iff f(v_0), \dots, f(v_n)$ span a simplex of L. Then the induced simplicial map is a homeomorphism g b/w $|K|$ and $|L|$.

g is called a simplicial homeomorphism.

☒

Δ^N is the complex consisting of an N-simplex and its faces. If K is a finite complex then $K \cong$ subcomplex of Δ^N.

$\mathbb{R}^J$, J arbitrary indexset.

$$\mathbb{R}^J = \{ f : J \to \mathbb{R} \} = \{ (x_\alpha)_{\alpha \in J} \}$$

$E^J \subseteq \mathbb{R}^J$ consisting of points $(x_\alpha)_{\alpha \in J}$ s.t.
$x_\alpha = 0$ for all but finitely many α's $\in J$.
generalized Euclidean space

$$|x - y| = \max \{ |x_\alpha - y_\alpha| \}_{\alpha \in J}$$

notion of simplex & simplicial complex K generalizes to $E^J \rightsquigarrow$ infinite dimensional simplicial complex K.

Abstract simplicial complexes

Defn An abstract simplicial complex is a collection $\mathcal{S}$ of finite non-empty sets st if $A \in \mathcal{S}$, then so is every non-empty subset of A.

$$\mathcal{S} \rightsquigarrow K$$

elements of $\mathcal{S} \rightsquigarrow$ simplices $\sigma \in K$

2nd part of def $\rightsquigarrow$ 2nd cond. in the def. of K.

$A \in \mathcal{S}$ is called a simplex of $\mathcal{S}$.

$\dim(A) = |A| - 1.$

Each nonempty subset of A is called a face of A.

$\dim(\mathcal{K})$ is the largest dim of one of its simplices.

$\dim(\mathcal{K}) = \infty$ if there is no such number.

vertex set V of $\mathcal{K}$ = union of all the one-point elements of $\mathcal{K}$.

$v \in V$ = 0-simplex $\{v\} \in \mathcal{K}$.

A subcollection of $\mathcal{K}$ that is itself a complex is called a subcomplex of $\mathcal{K}$.

$\mathcal{K} \cong \mathcal{T}$ if $\exists$ a bijective correspondence f mapping the vertex set of $\mathcal{K}$ to the vertex set of $\mathcal{T}$ s.t. $\{a_0, a_1, \dots, a_n\} \in \mathcal{K}$

$\iff \{f(a_0), f(a_1), \dots, f(a_n)\} \in \mathcal{T}.$

$\{v_0, v_1\} \in \mathcal{K}$

$\{v_0\} \in \mathcal{K}$

$\{v_1\} \in \mathcal{K}$