Lecture 21

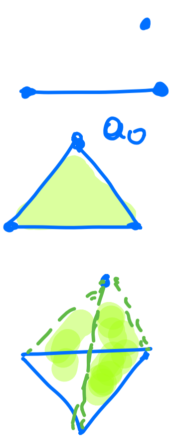
Please complete the Evaluation for the course on Moodle.

Recall :-

$\{a_0, a_1, \dots, a_n\}$ geom. ind. set in $\mathbb{R}^N$ then n-simplex σ spanned by $a_0, \dots, a_n$ is the set of points $x \in \mathbb{R}^n$ w.r.t

$$x = \sum_{i=0}^{n} t_i a_i \qquad w/ \ \sum t_i = 1. \quad , t_i \geq 0 \quad \forall i.$$

t_i's are called the barycentric coordinates of x w.r.t. $a_0, \dots, a_n$.



0-simplex

1-simplex

2-simplex

3-simplex

Properties :-

1) The barycentric coordinates $t_i(x)$ w.r.t. $a_0, \dots, a_n$ are continuous function of x.

Suppose
$$\begin{array}{ccc} x_n & \longrightarrow & x \\ \downarrow & & \downarrow \\ (t_i^n) & \longrightarrow & (t_i) \end{array}$$

$\sum t_i = 1$

$$\|x_n - x\| \to 0 \implies \left\| \sum t_i^n a_i - \sum t_i a_i \right\| \to 0$$

$$\leq \sum_{i=0}^{n} \|(t_i^n - t_i) a_i\| \to 0$$

$$\implies t_i^n \to t_i \quad \forall i$$

$\implies$ $t_i(x)$ are cont. functions of x.

2) σ equals the union of all line segments joining a_0 to points of the simplex s spanned by $a_1, \dots, a_n$.

3) σ is compact, convex set in $\mathbb{R}^n$.

4) Given a simplex σ, there is one and only one geom. ind. set of points $\{a_0, \dots, a_n\}$. spanning σ.

The points $a_0, a_1, \dots, a_n$ — vertices

n - dimension of σ

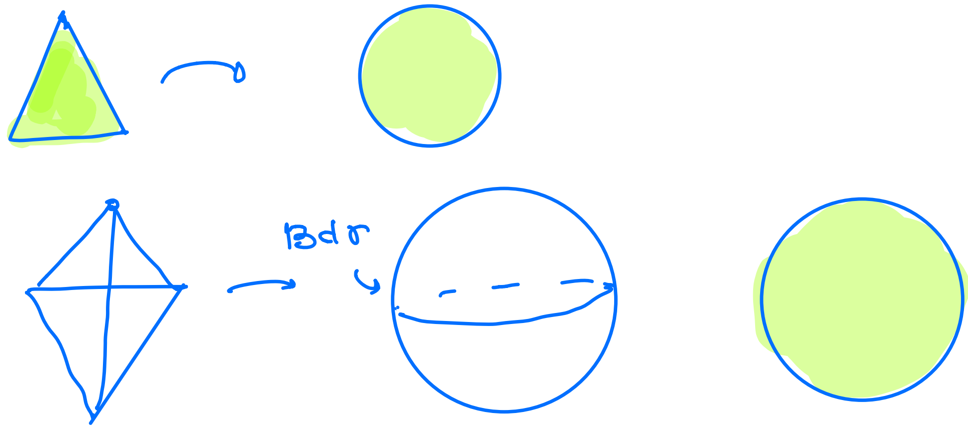
Any simplex spanned by a subset $\{a_0, \dots, a_n\}$ is called a face of σ.

union of all proper faces = boundary of σ

$\mathrm{Bd}\,\sigma$ or $\partial\sigma$

interior of σ $\quad \mathrm{Int}\,\sigma = \sigma - \mathrm{Bd}\,\sigma$

5) σ is n-simplex then there is a homeomorphism of σ w/ the unit ball B^n that carries Bd σ to S^{n-1}!

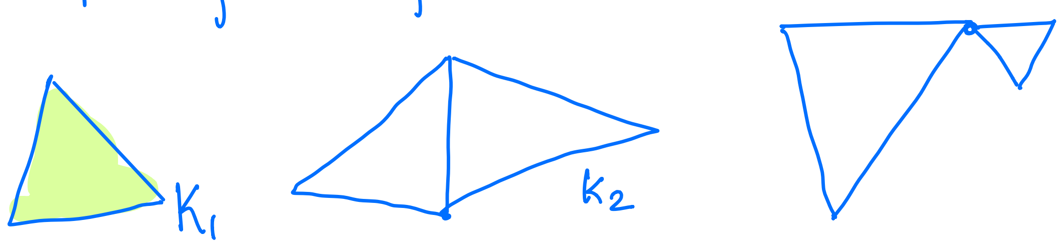


Complexes in $\mathbb{R}^n$

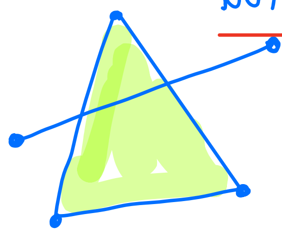
Def^n A simplicial complex K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ s.t.

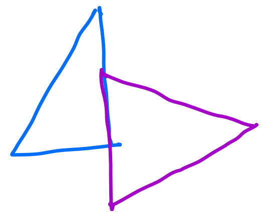
(1) Every face of a simplex of K is in K.

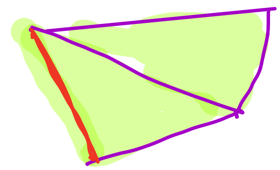
(2) The intersection of any two simplexes of K is a face of each of them.



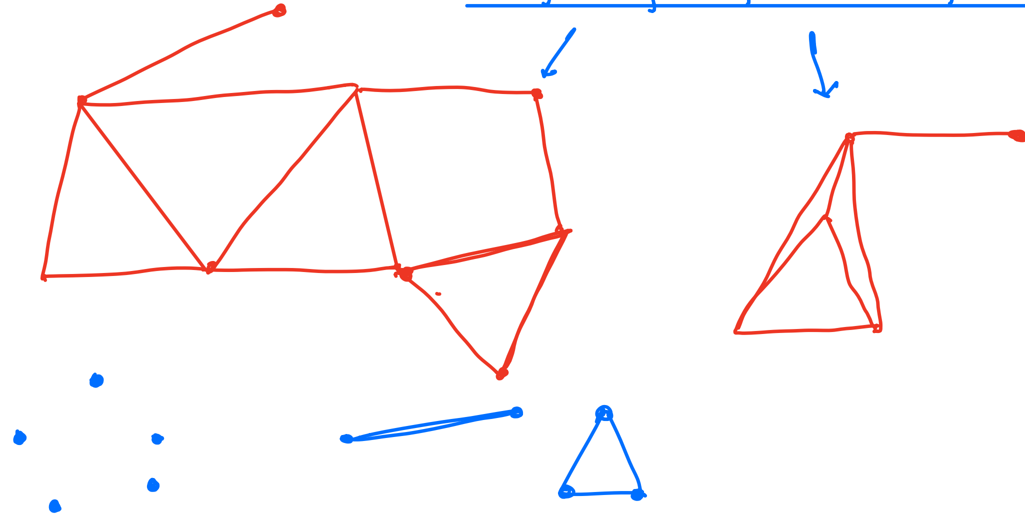
NOT a simplicial complex







Examples of simplicial complexes



Lemma:- A collection K of simplices is a simplicial complex $\iff$ the following hold:-

1) Every face of a simplex of K is in K.

2') Every pair of distinct simplices of K have disjoint interiors.

Proof:- Assume K is a simplicial complex.

Let σ and τ be two simplices K. We'll show that if $x \in \operatorname{int}\sigma \cap \operatorname{int}\tau \implies \sigma = \tau$.

Let $s = \sigma \cap \tau$. If s were a proper face of σ then $x \in \operatorname{Bd}\sigma \rightsquigarrow$ not possible as $x \in \operatorname{Int}\sigma$

$\therefore \quad s = \sigma$

Similarly $\quad s = \tau \quad \Rightarrow \quad \sigma = \tau$.

Conversely, assume 1) and 2) hold.

We'll show that if the set $\sigma \cap \tau$ is non-empty then it is equal to that face σ' of σ that is spanned by those vertices $b_0, b_1, \dots, b_m$ of σ that lies in τ.

Note that $\sigma' \subset \sigma \cap \tau$ b/c $\sigma \cap \tau$ contains $b_0, \dots, b_m$.

Want:- $\sigma \cap \tau \subset \sigma'$

Let $x \in \sigma \cap \tau \Rightarrow x \in \operatorname{int} s \cap \operatorname{int} t$ where s and t are some faces of σ and τ respectively.

$\therefore$ by 2) $\quad s = t$.

$\Rightarrow$ the vertices of s lie in τ

$\Rightarrow$ by def^n vertices of $s \subset \{b_0, \dots, b_m\}$

$\Rightarrow \quad s$ is a face of $\sigma' \Rightarrow x \in \sigma'$

$\Rightarrow \quad \sigma' = \sigma \cap \tau \quad \Rightarrow$ the def^n of K and the condition in Lemma are equivalent.

Def^n:- If L is a subcollection of a simplicial complex K that contains all faces of its elements

then L is a simplicial complex in its own right.
L is called a subcomplex of K.
One subcomplex of K is the collection of all simplices in K of dim at most $p \rightsquigarrow$ p-skeleton of K and is denoted by $K^{(p)}$.

The points of $K^{(0)}$ are called vertices of K.

Def^n:- Let $|K|$ be the subset of $\mathbb{R}^N$ that is the union of the simplices of K. Giving each simplex its natural topology as a subspace of $\mathbb{R}^N$, we can put a topology on $|K|$; we declare a subset A of $|K|$ to be closed in $|K| \iff A \cap \sigma$ is closed in σ $\forall$ simplex σ in K.

$|K|$ is called the underlying space of K or the polytope of K.

A space that is the polytope of a simplicial complex is called a polyhedron.

In general, the topology $|K|$ is finer than the topology $|K|$ has as a subspace of $\mathbb{R}^N$.

Ex: K is the collection of all 1-simplices in $\mathbb{R}$

of the form $[m, m+1]$, $m \in \mathbb{Z} \setminus \{0\}$ along with all simplices of the form $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, $n \in \mathbb{N}$.

Polytope of K is $\mathbb{R}$ as a set but not as a top. space.

$\mathbb{R}$

$\{1/n\}$ is closed in $|K|$ but not in $\mathbb{R}$.

ex. $K = \{\sigma_1, \sigma_2, \dots\}$ σ_i is a 1-simplex $\forall i$

σ_i is the 1-simplex in $\mathbb{R}^2$ w/ vertices 0 and $(1, 1/i)$

intersection is closed in $|K|$ but the intersection is not closed in the subspace top. of $|K|$ b/c $(0,0)$ is a limit point.