Lecture 20

- Problem Set 7 will be uploaded after the prob. session.

Recall:-

Thm.:- Let $X = P/\sim$, n edges labelled by letters $a_1, a_2, \ldots, a_n$. G is the set of all letters which appear in this list $\forall$ $i = 1, \ldots, n$, we write $p_i = 1$ if the arrow at edge i point counterclockwise around the boundary, $p_i = -1$ otherwise. Then

$$\pi_1(X) \cong \{ G \mid a_1^{p_1} \cdot a_2^{p_2} \ldots a_n^{p_n} = e \}.$$

$\pi_n(X)$ – n-th fundemental group – n-th homotopy group

$$\Sigma_g \not\cong \Sigma_h \rightsquigarrow \pi_1(\Sigma_g) \not\cong \pi_1(\Sigma_h).$$

$\mathbb{R}^1 \not\cong \mathbb{R}^n$ connectedness

$\mathbb{R}^2 \not\cong \mathbb{R}^n, n>2$ fundamental group – Poincaré 1901

$\mathbb{R}^n \not\cong \mathbb{R}^m, n \neq m$?

Poincaré conjecture 1903

Grigori Perelman 2003

Ricci flow

Thurston's Geometrization conjecture.

Betti — associated to every space X a sequence of abelian groups $\rightsquigarrow$ homology groups.

Homology Theory $\rightsquigarrow$ $(\text{boundary})^2 = \text{boundary} \circ \text{boundary}$
$= 0$

simplicial homology groups

singular hom. groups

Čech homology groups.

deRham cohomology (homology)

d - exterior derivative

Simplicial Homology

Def^n Let $\{a_0, \dots, a_n\}$ points in $\mathbb{R}^n$, this set is geometrically independent if for any $t_i \in \mathbb{R}$ the equations

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i a_i = 0$$

$$\Rightarrow \quad t_0 = t_1 = \cdots = t_n = 0.$$

$\{a_0\}$ - geometrically ind.

$\{a_0, \dots, a_n\}$ g.i. $\Leftrightarrow$ $\{a_1 - a_0, a_2 - a_0, \dots, a_n - a_0\}$ linearly independent.

→ $\{a_0, \ldots, a_n\}$ geom. ind. , we define the n-plane P spanned by these points as the set of $x \in \mathbb{R}^n$ s.t

$$x = \sum_{i=0}^{n} t_i a_i \quad , \; t_i \in \mathbb{R} \text{ s.t } \sum t_i = 1.$$

An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which is a composition of a translation $\left(T(x) = x + p \, , \; p \in \mathbb{R}^n \text{ fixed}\right)$ and a non-singular linear transformation.

If T is an affine trans. then T takes geom. ind. sets to geom. ind. sets , $P = \langle a_0, a_1, \ldots, a_n \rangle$

$T(P) = \langle Ta_0, \ldots, Ta_n \rangle$.

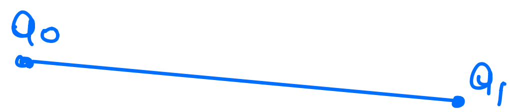
Defn:- Let $\{a_0, \ldots, a_n\}$ is a geometrically independent set in $\mathbb{R}^n$. We define the n-simplex σ spanned by $a_0, \ldots, a_n$ to be the set of all points $x \in \mathbb{R}^n$ s.t.

$$x = \sum_{i=0}^{n} t_i a_i \quad \text{w/} \quad \sum_{i=0}^{n} t_i = 1 \, , \; t_i \geq 0 \;\; \forall i.$$

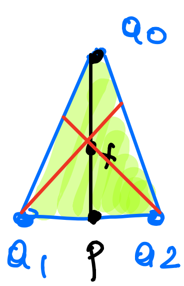
The numbers t_i are uniquely determined by x. They are called the barycentric coordinates of the point $x \in \sigma$ w.r.t. $a_0, \ldots, a_n$.

Ex: 0-simplex. $\rightarrow$ point.

1-simplex $\rightsquigarrow$ $\langle a_0, a_1 \rangle = \{x \mid x = t a_0 + (1-t) a_1, \ 0 \le t \le 1\}$



2-simplex $= \langle a_0, a_1, a_2 \rangle$ = triangle having three three points as vertices.

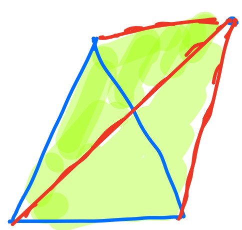


$\Rightarrow x \neq a_0$

$$x = \sum_{i=0}^{2} t_i a_i = t_0 a_0 + (1-t_0) \underbrace{\left[\left(\frac{t_1}{\lambda}\right) a_1 + \left(\frac{t_2}{\lambda}\right) a_2 \right]}_{\text{p lying on the line seg. joining } a_1, a_2}$$

$$(\lambda = 1 - t_0)$$

3-simplex $= \langle a_0, a_1, a_2, a_3 \rangle$
= tetrahedron



Properties of simplices

1) The barycentric coordinates $t_i(x)$ of x w.r.t. $a_0, \ldots, a_n$ are continuous functions of x.

2) σ is the union of all line segments joining a_0 to points of the simplex spanned by $a_1, \ldots, a_n$.
Two such line segments intersect only in the point a_0.

3) σ is a compact, convex set in $\mathbb{R}^n$.

$$\sigma = \bigcap_{\substack{C \text{ convex set in } \mathbb{R}^n \\ a_0, \ldots, a_n \in C}} C$$

4) Given σ there is one and only one geom. ind. set of points spanning σ.

$\sigma = \langle a_0, \ldots, a_n \rangle$

$a_0, a_1, \ldots, a_n$ — vertices of σ

n — dimension of σ

any simplex spanned by a subset of $\{a_0, \ldots, a_n\}$ — face of σ.

— face of σ spanned by $\langle a_1, \ldots, a_n \rangle$
— face opposite a_0.

proper faces — faces of σ different from σ itself.

→ union of all the proper faces of σ is called the boundary of σ $\partial\sigma$, $\mathrm{Bd}\,\sigma$

interior of σ , $\operatorname{int} \sigma = \sigma - \operatorname{Bd} \sigma$.

↳ open simplex.