Lecture 2

* Course outline updated w/ info about problem sets submission and its advantages.

* Recordings will be available.

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Metric Spaces (examples of topological spaces)

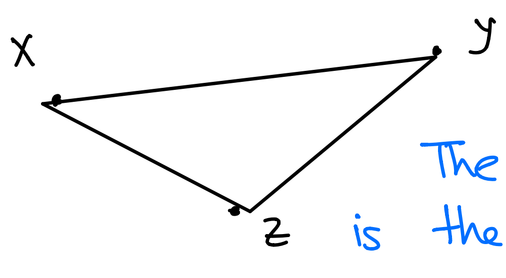
Def^n Let X is a non-empty set. Then (X, d) is a metric space if $d : X \times X \to \mathbb{R}$ is a function which satisfies, $\forall\, x, y, z \in X$

i) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$.

ii) $d(x,y) = d(y,x)$ (symmetry)

iii) $d(x,y) \leq d(x,z) + d(y,z)$

(Triangle inequality)



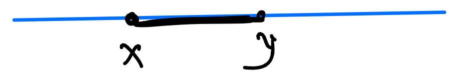
d is a metric on X.

The topology generated by d is the metric topology on X.

Examples

① $(\mathbb{R}, |\cdot|)$ $\quad x, y \in \mathbb{R}$

$d(x,y) = |x-y|$



$$d(x,y) \leq d(x,z) + d(y,z)$$

$$|x-y| = |x+z-z-y|$$

$$\leq |x-z| + |y-z| = d(x,z) + d(y,z)$$

② $\mathbb{C}$, $\quad z = x+iy$, $\quad |z| = \sqrt{x^2+y^2}$

Then $d(z,w) = |z-w|$

$(\mathbb{C}, d)$ a metric space.

③ $\mathbb{R}^n$, $\quad x = (x_1, x_2, \ldots, x_n)$
$\quad y = (y_1, y_2, \ldots, y_n)$ $\in \mathbb{R}^n$

$$d_1(x,y) = \sum_{k=1}^{n} |x_k - y_k|$$

$$d_\infty(x,y) = \max_{1\leq k\leq n} \{ |x_k - y_k| \}$$

$$d_2(x,y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$$

(4) $C([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f$ is continuous $\}$

$$d_\infty(f,g) = \sup |f-g|$$

Exercise :- Check that all the above are indeed metric spaces.

(5) Suppose $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space.

Define norm of $x \in V$ as $\|x\| = \sqrt{\langle x, x \rangle}$

This gives rise to a metric on V

$x, y \in V$

$$d(x,y) = \|x-y\|$$

$(V, \|\cdot\|)$ is a normed linear space and is a metric space.

Every IPS is a metric space where the metric comes from a norm.
But the converse is NOT true. (PSet 1)

⑥ Discrete metric on X.

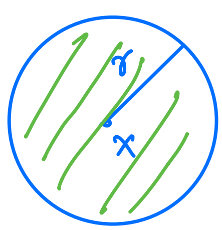
$$d(x,y) = \begin{cases} 0 & \text{if } y = x \\ 1 & \text{otherwise} \end{cases}$$

It is a metric on X generating the discrete topology on X.

Open sets & Open balls

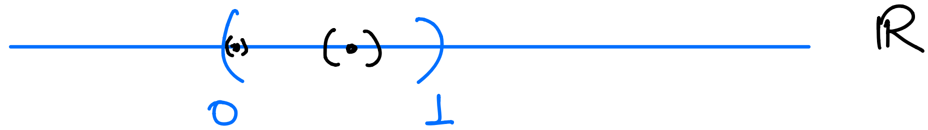
(X,d) metric space, then an open ball of radius r, centred at $x \in X$

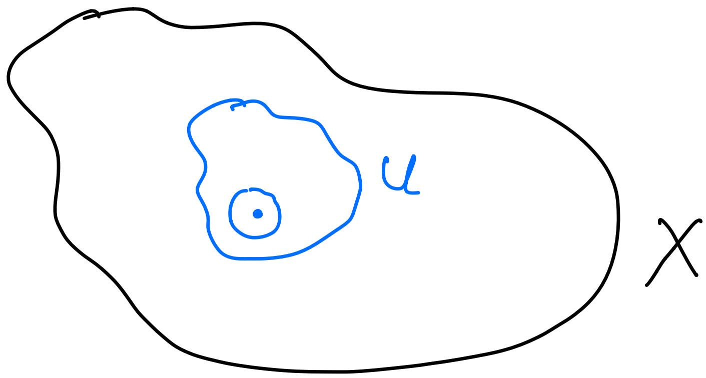
$$B_r(x) = \{ y \in X \mid d(x,y) < r \}$$



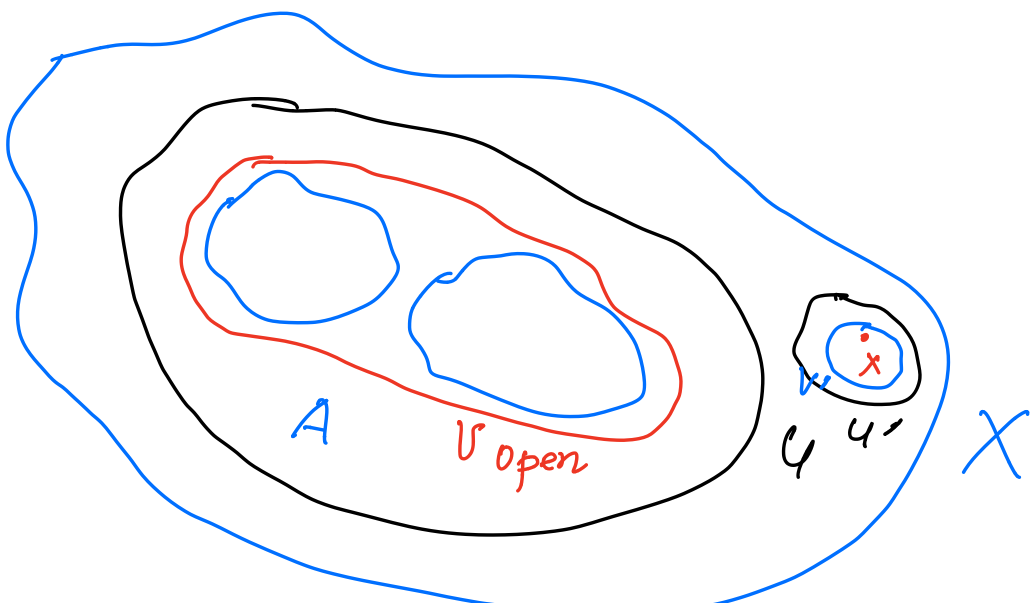
$B_r(x)$

$U \subset X$ is an open set if $\forall\, x \in U\ \exists$ $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$.





Suppose $A \subset X$. Then U is said to be neighbourhood of A if $\exists$ an open set V in X s.t. $A \subset V \subset U$.

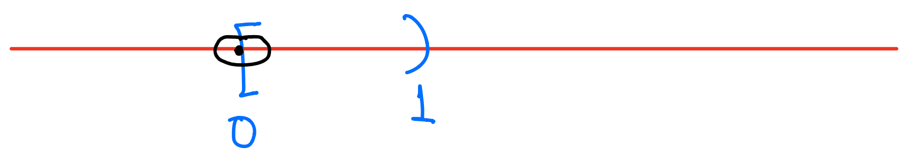


$\{x\} \subset X$

Closet Sets

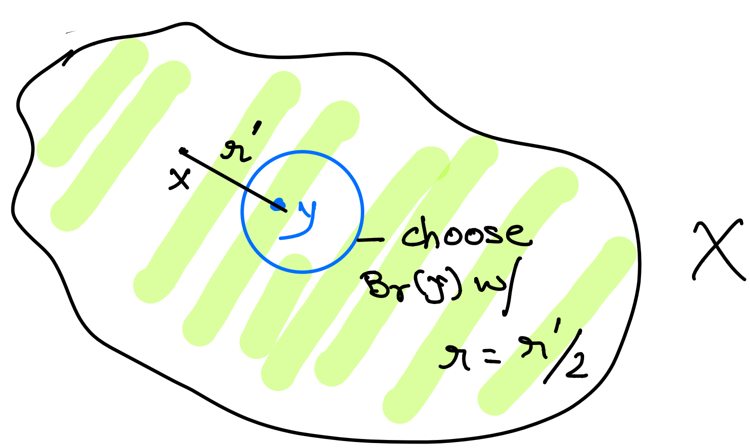
$A \subset X$ is closed set if $X \setminus A$ is an open set.

Remark :- If a set is NOT closed then it is not necessarily open.



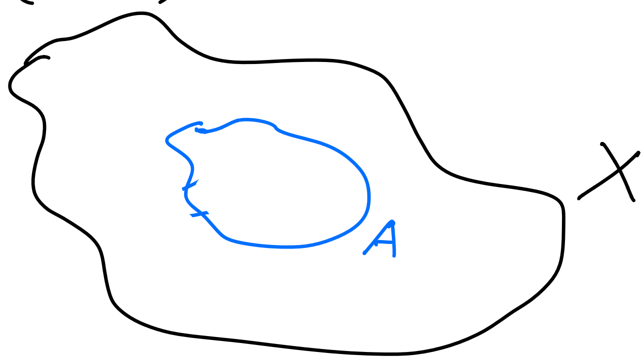
Exercise Check whether $\{x\}$ is an open set or closed set or both or neither wher (X, d), d is the discrete metric.

In general, $\{x\}$ in (X, d) is always closed.



Convergence of sequences in a metric space

(X, d) and $A \subset X$

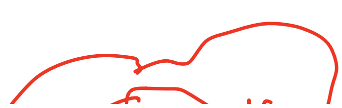


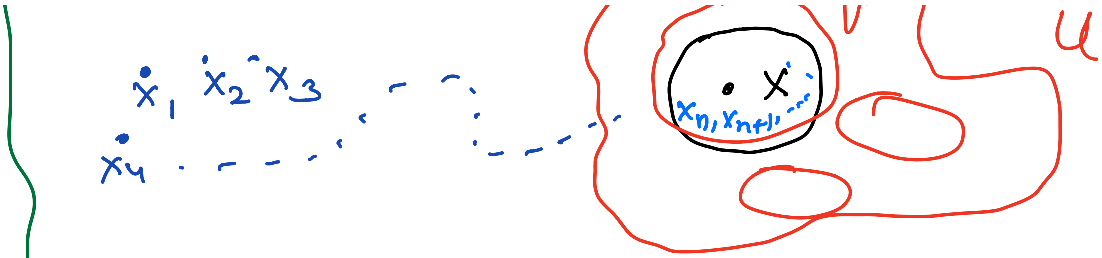
restriction of d from X to A makes (A, d_A) a metric space.

Defn In (X, d) a sequence (x_n) in X converges to $x \in X$ if $\forall\, \epsilon > 0$ $x_n \in B_\epsilon(x)$ $\forall\, n$ sufficienty large.

Equivalently, $\forall$ neighbourhood U of x $x_n \in U$ $\forall\, n$ sufficiently large.

$$\lim x_n = x \quad \text{or} \quad x_n \longrightarrow x$$





This def^n doesn't require a metric if we can make sense of what a "neighbourhood" is.

Continuous function

(X, d_X) and (Y, d_Y) metric spaces.

$f: X \to Y$ is continuous if any of the following equivalent conditions hold:-

1. $\forall s \in X$ and $\forall \epsilon > 0 \; \exists \; \delta > 0$ s.t. if $d_X(s,t) < \delta \implies d_Y(f(s), f(t)) < \epsilon$.

2. For every open set $U \subset Y$, the preimage $f^{-1}(U) = \{ x \in X \mid f(x) \in U \}$ must be open in X.

③ If $x_n \to x$ in X $\Rightarrow$ $f(x_n) \to f(x)$ in Y.

① $\Rightarrow$ ② exercise.

② $\Rightarrow$ ③

2) $\Rightarrow$ 3)

$x_n \to x$ given. Want:- $f(x_n) \to f(x)$

Suppose U is a nbd of $f(x)$. $\Rightarrow$ $\exists$ an open set $V \subset U$ s.t. $f(x) \in V$

$\Rightarrow$ $f^{-1}(U) \supset f^{-1}(V) \ni x$.

open as we are assuming ②.

$f^{-1}(U)$ is a nbd of x

$x_n \in f^{-1}(U)$ for n sufficiently large

$\Rightarrow$ $f(x_n) \in U$ " —— "

$\Rightarrow$ $f(x_n) \to f(x)$.

Remark We didn't use the metric at all.

(3) $\Rightarrow$ ②

We'll prove the contrapositive

$\Rightarrow$ $\sim$② $\Rightarrow$ $\sim$③

$\exists$ an open set U in Y s.t. $f^{-1}(U)$ is not open in X.

$\Rightarrow$ $\exists$ $x \in f^{-1}(U)$ s.t no open ball around x is contained in $f^{-1}(U)$.

$\Rightarrow$ $\forall$ $n \in \mathbb{N}$ $\exists$ $x_n \in B_{1/n}(x)$

s.t. $x_n \notin f^{-1}(U)$.

$\Rightarrow$ $f(x_n) \notin U$. — 1

$x_n \to x$ by the way we chose them.

but $f(x_n)$ can never converge to $f(x)$. b/c U is a nbd of $f(x)$ and it doesn't contain any $f(x_n)$.

<u>Homeomorphism</u> $f: X \to Y$ is a homeomorphism if f is continuous, bijective and $f^{-1}: Y \to X$ is continuous.

Homeomorphism is an equivalence relation.

$X \cong Y$

Exe. Consider $(\mathbb{R}^n, d_2)$. Then

$$(B_r(x), d_2) \cong (\mathbb{R}^n, d_2).$$

Any two balls in $\mathbb{R}^n$ are homeomorphic.