Lecture 19

Recall:- $\{G_\alpha\}_{\alpha\in J}$ collection of groups

$\underset{\alpha\in J}{*} G_\alpha$ - free product of groups $\longrightarrow$ collection of all reduced words in $\{G_\alpha\}_{\alpha\in J}$.

→ S is a set, the free group on S

$$F_S = \underset{\alpha\in S}{*} \mathbb{Z}$$

$\parallel$

set of all reduced words $a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$,

$n \geq 0$, $p_i \in \mathbb{Z}$, $p_i \neq 0$, $a_i \in S$ w/ $a_i \neq a_{i+1}$ $\forall i$.

elements of S are called generators of F_S.

→ S is a set, a relation in S means any eqn. of the form "$a = b$", $a, b \in F_S$.

- S is a set, R is a set consisting of relations in S, we define the group

$$\{S \mid R\} = F_S / \langle R' \rangle_N$$

R' → set of all elements of the form $a^{-1}b \in F_S$ for relation "$a=b$" in R.

$$[w] = [w'] \iff w^{-1}w' \in \langle R' \rangle_N .$$
$$\Updownarrow$$
$$\text{"} w = w' \text{"} \in R .$$

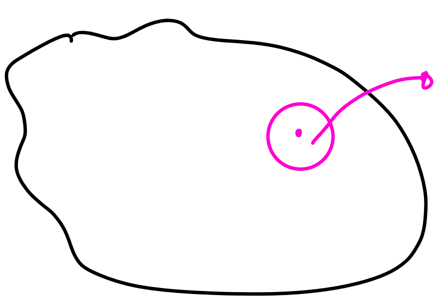
$G \cong \{S \mid R\}$ – presentation of G.

If $|S|, |R| < \infty$ then G is finitely presented.

$F_{\{a\}} \cong \mathbb{Z}$, $\{a \mid a^p = e\} \cong \mathbb{Z}_p$

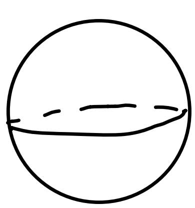
$\{a, b \mid ab = ba\} \cong \mathbb{Z} \times \mathbb{Z}$.

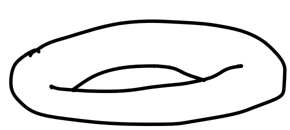
Fundamental group of surfaces



look like an open subset of $\mathbb{R}^2$

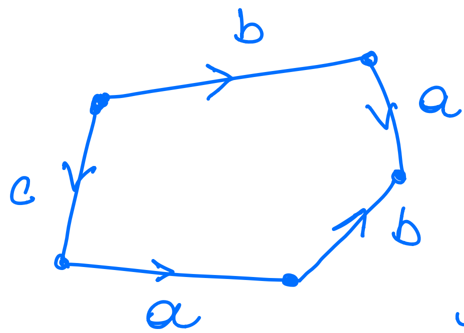
X → surface.





$\mathbb{RP}^2$

We'll consider polygons



$P \subset \mathbb{R}^2$

compact, convex region in $\mathbb{R}^2$

Suppose P is bounded by n edges.

edges $a_1, a_2, \dots, a_n$, arrows on edges.

We define a topological space

$$X = P/\sim \quad - \text{ surface}$$

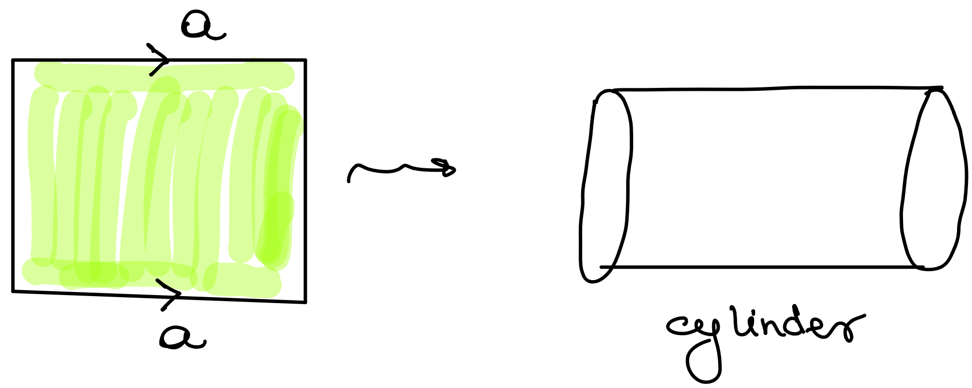
$\sim$ - trivial on the interior of P.

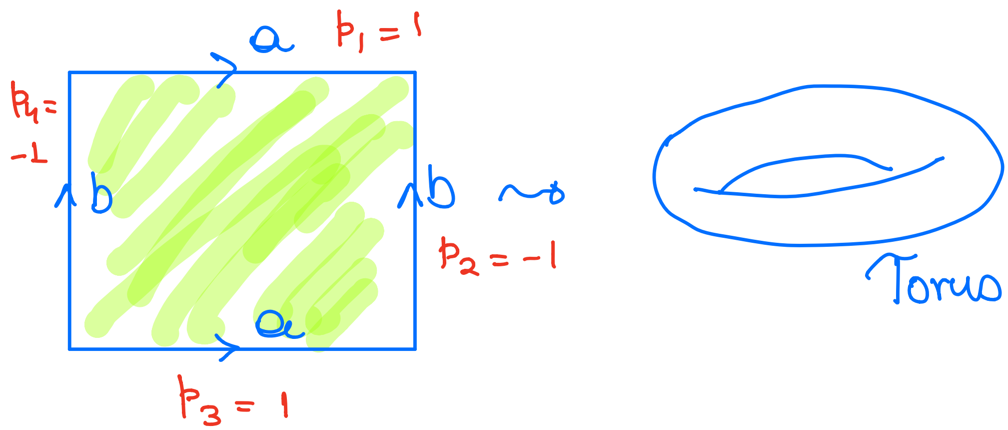
$\sim$ on the boundary is as follows:-

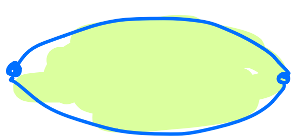
identify all the vertices to a single point

identify any pair of edges labelled by the same letter via a homeomorphism which should match the direction of arrows.

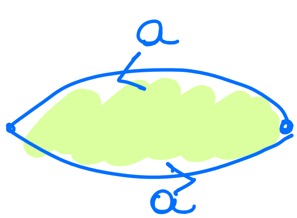
Fact :→ All compact surfaces can be presented as a quotient of a polygon.







$\cong \mathbb{D}^2$



every point on $\partial \mathbb{D}^2$ is being identified w/ its antipodal point. $\rightarrow \mathbb{RP}^2$

Theorem :-

Suppose $X = P/\sim$ is a space as described above, P has n edges labelled by $a_1, a_2, \ldots, a_n$, listing them in the order in which they appear as the ∂P is traversed once counterclockwise.

Let G denote the set of all letters that appear in the list and $\forall\ i = 1, \ldots, n$ we write

$P_i = 1$ — if the arrow at edge i points counterclockwise around the boundary

$P_i = -1$ — " ——— " ——— "
clockwise " ——— "

Then $\pi_1(X)$ is isomorphic to the group w/ generators G and exactly one relation

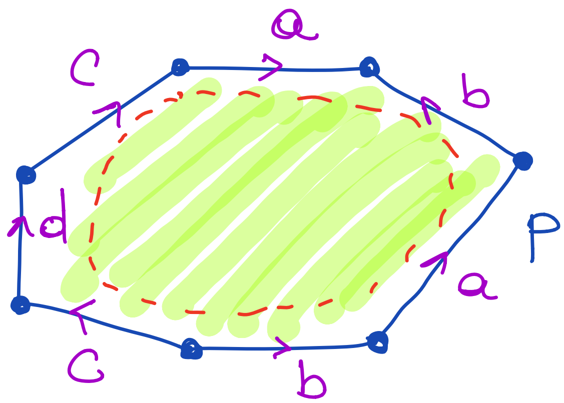
$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} = e \text{, i.e,}$$

$$\pi_1(X) \cong \{ G \mid a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} = 1 \}.$$

Proof:- Let $P^{\perp} = \partial P/\sim \subset X$.

$\because$ all vertices are identified to a point

$\Rightarrow$ P' is homeomorphic to wedge sum of circles, one for each of the letters that appear as labels of the edge.



$G = \{a, b, c, d\}$

By the Seifert-van Kampen theorem

$$\pi_1(P') \cong \pi_1(S^1) * \pi_1(S^1) * \cdots * \pi_1(S^1)$$
$$\cong \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} = F_G .$$

decompose $X = A \cup B$, $A, B \underset{\text{open}}{\subseteq} X$

A = interior of P

B = open nbd of P'

$A \cap B$ is homeomorphic to an annulus $S^1 \times (-1,1)$

$\Rightarrow$ if $p \in A \cap B$, $\pi_1(A \cap B, p) \cong \mathbb{Z}$

$\because$ $A \cong D^2 \Rightarrow \pi_1(A) = 0$

B deformation retracts to P'

$\Rightarrow \pi_1(B,p) \cong \pi_1(P') \cong F_G$.

$\therefore$ By the van Kampen thm,

$\pi_1(X,p)$ is a quotient of $\pi_1(A) * \pi_1(B) \cong F_G$.

$\cong$

$F_G /$ normal sbgp. generated by the relation

that if $j_A : A\cap B \hookrightarrow A$

$j_B : A\cap B \hookrightarrow B$

$\underbrace{(j_A)_*[\gamma]}_{\text{trivial as } \pi_1(A)=0.} = (j_B)_*[\gamma]$, $[\gamma] \in \pi_1(A\cap B, p) \cong \mathbb{Z}$.

$(j_B)_*[\gamma] \in \pi_1(B,p)$

$\downarrow$ becomes the concatenated loop

$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$. $\therefore$ the relation is

$(j_B)_*[\gamma] = e \Rightarrow$ the relation is

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} = e$$

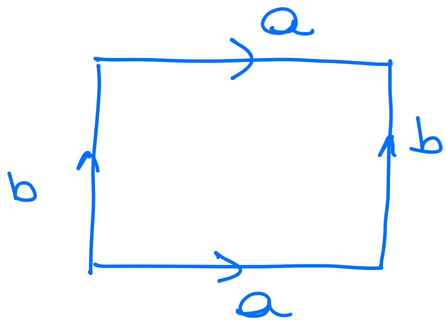
$\Rightarrow$ by von Kampen theorem

$$\pi_1(X) = \frac{\pi_1(A) * \pi_1(B)}{\{a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} = e\}} = \{G \mid a_1^{p_1} \cdots a_n^{p_n} = e\}$$

$\boxed{//}$

<u>Examples</u> :-

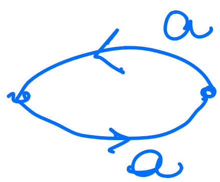
① $\mathbb{T}^2$ $\quad \pi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$



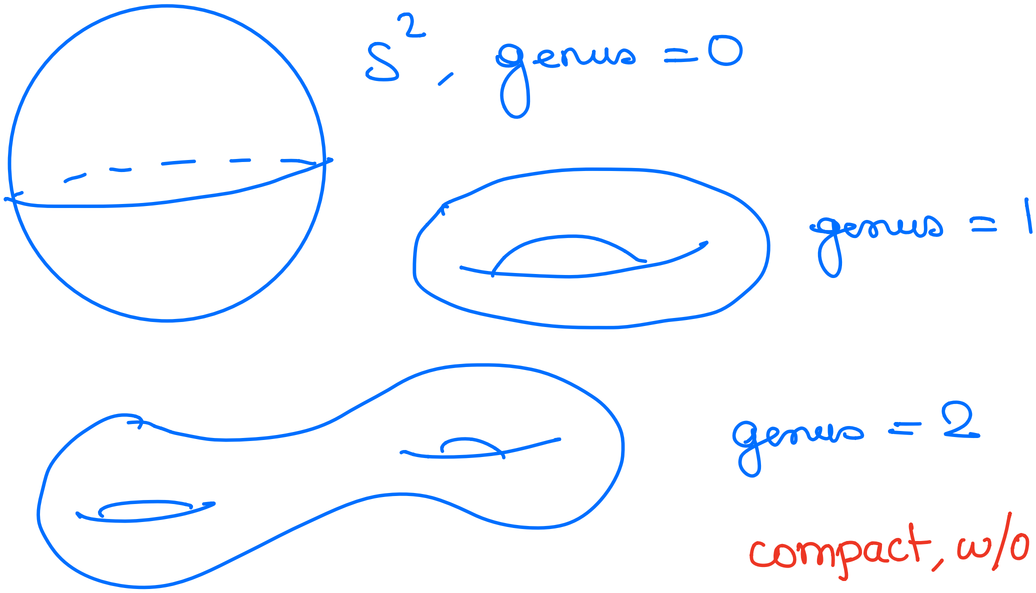
$G = \{a, b\}$

$$\pi_1(\mathbb{T}^2) \cong \{a, b \mid b^{-1} a^{-1} b a = e\}$$
$$= \{a, b \mid ab = ba\}$$
$$= \mathbb{Z} \times \mathbb{Z}$$

② $\mathbb{RP}^2$ $\quad \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$



$$\rightsquigarrow \pi_1(\mathbb{RP}^2) = \{a \mid a^1 a^1 = e\}$$
$$= \{a \mid a^2 = e\} \cong \mathbb{Z}_2.$$



compact, w/o boundary

Def^n:- For any integer $g \geq 0$, the closed orientable surface Σ_g of genus g is defined to be S^2 if $g = 0$ and otherwise $\Sigma_g = P/\sim$, P is a polygon w/ $4g$ edges labelled by $2g$ distinct letters $\{a_i, b_i\}_{i=1}^{g}$ in the order

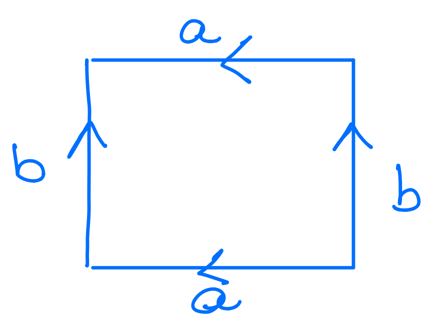
$$a_1, b_1, a_1, b_1, a_2, b_2, a_2, b_2, \ldots, a_g, b_g, a_g, b_g$$

s.t. the arrows point counterclockwise on the first appearance of each letter in the sequence and clockwise on the second appearance.

e.g. $\mathbb{T}^2$ is a surface of genus 1.

$\downarrow$ polygon $4 \cdot 1 = 4$ sides

2 distinct letters



Exercise:- Try to write down $\pi_1(\Sigma_g)$.

In particular, use this description to solve Prob. on $\pi_1(\mathbb{T}^2 \# \mathbb{T}^2)$.

Proof of the van Kampen theorem

Thm:- Suppose $X = \bigcup_{\alpha \in J} A_\alpha$, $A_\alpha \underset{\text{open}}{\subset} X$
path-connected

$i_\alpha : A_\alpha \hookrightarrow X$ and $j_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$

$\alpha, \beta \in J$. Let $p \in \bigcap_{\alpha \in J} A_\alpha$.

① If $A_\alpha \cap A_\beta$ is path-connected $\forall$ pair $\alpha, \beta \in J$ then

$$\Phi : \underset{\alpha \in J}{*} \pi_1(A_\alpha, p) \longrightarrow \pi_1(X, p) \text{ s.t.}$$

$$\Phi|_{\pi_1(A_\alpha, p)} = (i_\alpha)_*,$$ is surjective.

(Already proved for the special case of the vanKampen thm.)

② If $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected $\forall$ triple $\alpha, \beta, \gamma \in J$ then

$$\operatorname{Ker} \Phi = \langle S \rangle_N \quad \text{where}$$

$$S = \left\{ (j_{\alpha\beta})_* [\gamma] \left((j_{\beta\alpha})_* [\gamma] \right)^{-1} \,\middle|\, \alpha, \beta \in J, \; [\gamma] \in \pi_1(A_\alpha \cap A_\beta, p) \right\}.$$

So, if $F = \underset{\alpha \in J}{*} \pi_1(A_\alpha, p)$, then

$$\pi_1(X, p) \cong F / \langle S \rangle_N .$$

<u>Proof</u>:- Want to prove:-

$\Phi(w) = \Phi(w')$ for reduced words $w, w' \in F$. then $[w] = [w']$ in $F/\langle S \rangle_N$. $\}$ – ①

γ is a lopp based at p in X, we say

$[\gamma]$ can be factored in the following sense

$$[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n] \quad \text{s.t.}$$

$[\gamma_i]$ is a loop based at p and is contained in A_{α_i}.

We know that $[\gamma]$ can be factored.

Any factorization of $[\gamma]$ $\rightsquigarrow$ reduced word $w \in F$, $w = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n]$.

Also $\Phi(w) = [\gamma]$.

Conversely, $w \in \Phi^{-1}([\gamma])$ can be realized as a factorization of $[\gamma]$ s.t. each letter is a loop based at p and contained in exactly one of the open sets.

$\therefore$ Showing 1 is same as this:- we need to show that any two factorizations of γ can be related to each other by a finite sequence of the following operations and their inverses.

① If γ_i and γ_{i+1} are adjacent loops, i.e, $\alpha_i = \alpha_{i+1}$, we replace them w/ $\gamma_i * \gamma_{i+1}$.

② replace some γ_i w/ γ_i' s.t. $\gamma_i \simeq_p \gamma_i'$ in A_{α_i}.

③ If $\gamma_i \in A_{\alpha_i} \cap A_\beta$, $\alpha_i, \beta \in J$, then we can replace α_i w/ β i.e, in the corresponding reduced word in F.

whenever we have

$(j_{\alpha_i \beta})_* [\gamma_i] \in \pi_1(A_{\alpha_i}, p)$, we can replace it w/ $(j_{\beta \alpha_i})_* [\gamma_i] \in \pi_1(A_\beta, p)$.

This operation ③ changes the reduced word $w \in F$, it won't change the eq. class $[w] \in F/\langle S \rangle_N$.

Basic idea:- create a subdivision of $I \times I$ w/ the required properties.

If $\gamma_1 * \gamma_2 * \cdots * \gamma_n \simeq_p \gamma_1' * \gamma_2' * \cdots * \gamma_{n'}'$

$\Rightarrow \exists \ H : I \times I \longrightarrow X$ w/

$H(0, \cdot) = \gamma_1 * \gamma_2 * \cdots * \gamma_n$

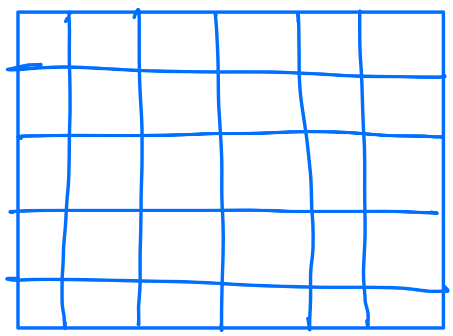
$H(1, \cdot) = \gamma_1' * \gamma_2' * \cdots * \gamma_{n'}'$

$H(s, 0) = H(s, 1) = p \quad \forall s \in I.$

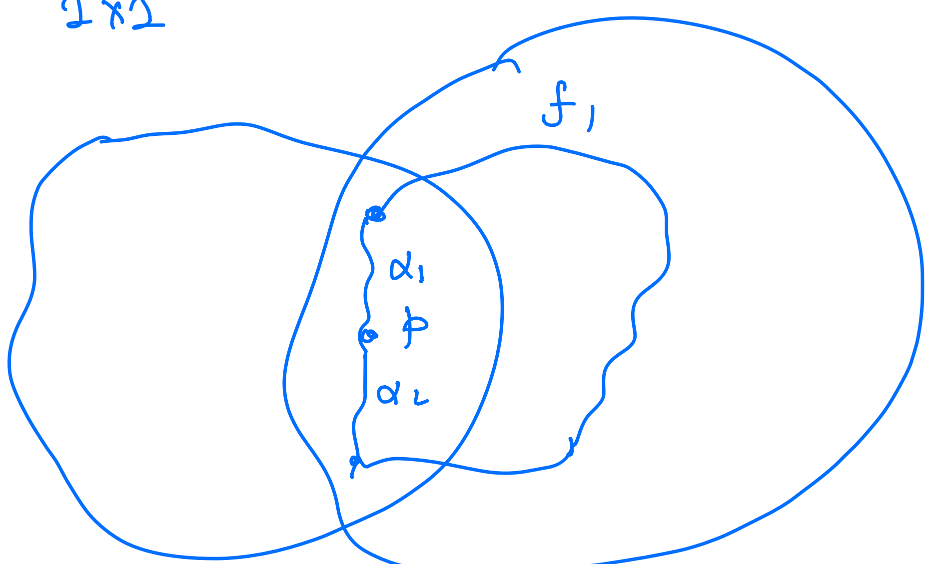
create a subdivision of $I \times I$ s.t.

$[s-2\epsilon, s+2\epsilon] \times [t-2\epsilon, t+2\epsilon] \in H^{-1}(A_\alpha)$

$\alpha \in J.$



$I \times I$



cf. Prof. Wendl's notes on the proof of the vanKampen.

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