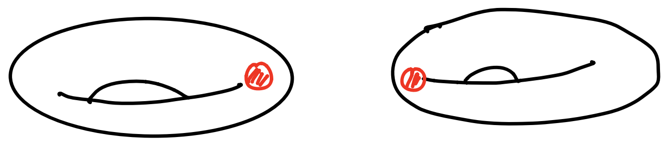
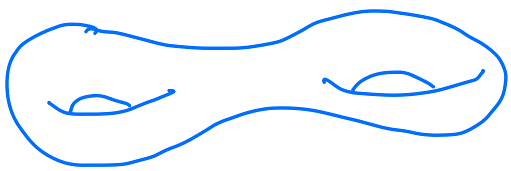
Lecture 18

- Pset 6 has been uploaded, due on 22nd June '21.

Fundamental group of $T \# T$ (mistake made in Lec. 17)



$\downarrow$



retracts onto fig. 8.

~~$\pi_1(\mathbb{T}^2 \# \mathbb{T}^2) \simeq$ free group on two generators.~~ Wrong.

$$j_* : \pi_1(\text{fig}(8)) \xrightarrow[\text{injective hom.}]{} \pi_1(\mathbb{T}^2 \# \mathbb{T}^2)$$

$\pi_1(\mathbb{T}^2 \# \mathbb{T}^2)$ has a subgroup that is isomorphic to $\pi_1(8) \simeq$ free group on two generators.

$\{G_\alpha\}_{\alpha \in J}$ generate G if $x \in G$ can be written as finite sum $\sum_{x_\alpha \in G_\alpha} x_\alpha$, $x_\alpha = e$ $\forall$ but finitely many α's.

If for each $x \in G$, the expression $x = \sum x_\alpha$ is unique then G is said to be a direct sum of the groups G_α

$$G = \bigoplus_{\alpha \in J} G_\alpha$$

Lemma Let G be an abelian group; let $\{G_\alpha\}$ be a family of subgroups of G. If $G = \bigoplus_{\alpha \in J} G_\alpha$ then G satisfy the following :-

Given any abelian group H and family of hom. $h_\alpha : G_\alpha \longrightarrow H$ $\exists!$ a hom. $h : G \longrightarrow H$ s.t. $h|_{G_\alpha} = h_\alpha$ $\forall \alpha$. —— (1)

Conversely, if the groups G_α generate G and condition (1) holds. then $G = \bigoplus_{\alpha \in J} G_\alpha$.

Free abelian groups

Defn G is an abelian group and let $\{a_\alpha\}$ be an indexed family of elements of G; $G_\alpha = \langle a_\alpha \rangle$.
If the groups G_α generate G, we say that the elements a_α generate G. If each G_α is infinite cyclic and if $G = \bigoplus_{\alpha \in J} G_\alpha$ then G is said to be free abelian group having the elements $\{a_\alpha\}$ as a basis.

Thm. If G is a free abelian group w/ basis $\{a_1, a_2, \dots, a_n\}$ then n is uniquely determined by G.

Proof. $G \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$

$2G \cong (2\mathbb{Z}) \times \cdots \times (2\mathbb{Z})$

$$G/2G \cong \underbrace{(\mathbb{Z}/2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/2\mathbb{Z})}_{\text{cardinality } 2^n}$$

$\therefore$ n is uniquely determined by G.

n is called the rank of the free abelian group G and is uniquely determined.

<u>Lemma</u>:- (Generalized form)

Suppose $X = \bigcup_{\alpha \in J} A_\alpha$ for collection of open subsets $\{A_\alpha\}_{\alpha \in J}$ s.t.

i) A_α is path-connected $\forall\ \alpha \in J$.

ii) $A_\alpha \cap A_\beta$ is path-connected for every pair $\alpha, \beta \in J$

iii) $\bigcap_{\alpha \in J} A_\alpha \neq \phi$ let $p \in \bigcap_{\alpha \in J} A_\alpha$.

Then $A_\alpha \xrightarrow{j_\alpha} X$ inclusion map. Then $\pi_1(X, p)$ is generated by $(j_\alpha)_* (\pi_1(A_\alpha, p)) \subset \pi_1(X, p)$.

<u>Def^n</u> Suppose $\{G_\alpha\}_{\alpha \in J}$ is a collection of groups w/ $e_\alpha \in G_\alpha$ the identity element. For any integer $n \geq 0$ w/ $b_1 b_2 \ldots b_n$ an ordered set alongwith a corresponding ordered set $\alpha_1, \alpha_2, \ldots \alpha_n \in J$ is called a word in $\{G_\alpha\}_{\alpha \in J}$ if $b_i \in G_{\alpha_i}$ $\forall\ i = 1, \ldots, n$.

Empty word:- $n=0$ $\leadsto$ will serve as the identity element later.

A word $a_1 a_2 \dots a_n$ is called a reduced word if

- none of the letters a_i are the identity element $e_{\alpha_i} \in G_{\alpha_i}$.
- no two adjacent letter a_i and a_{i+1} satisfy $\alpha_i = \alpha_{i+1}$, i.e., the groups that appear in adjacent positions are distinct.

Empty word trivially satisfies both the conditions $\Rightarrow$ its a reduced word.

$$\{\text{words}\} \xrightarrow{\text{reduction}} \{\text{reduced words}\}$$

<u>Defn</u> The free product $\underset{\alpha \in J}{*} G_\alpha$ of a collection of groups $\{G_\alpha\}_{\alpha \in J}$ is defined as the set of all reduced words in $\{G_\alpha\}_{\alpha \in J}$.

The product of two reduced words $w = b_1 b_2 \dots b_n$, $w' = b_1' b_2' \dots b_m'$ is the reduction of concatenated word

$ww' = b_1 \dots b_n b_1' b_2' \dots b_m'$.

The identity element is the empty word and we denote it $e \in \underset{\alpha \in J}{*} G_\alpha$.

$w^{-1} = b_n^{-1} b_{n-1}^{-1} \cdots b_2^{-1} \cdot b_1^{-1}$

$G_1 * G_2 * \cdots * G_n$.

Ex. ① Let $G_1 = G_2 \cong \mathbb{Z}_2$

a, b denote the nontrivial elements in G_1 and G_2 respectively.

$$\mathbb{Z}_2 * \mathbb{Z}_2 \cong G_1 * G_2 = \{e, a, b, ab, ba, aba, bab, abab, baba, \ldots\}$$

② $G_1 \cong \mathbb{Z}$, $G_2 \cong \mathbb{Z}_2$

$\quad\; \| \qquad\qquad \|$

$\langle a \rangle \qquad \langle b \rangle$

$g_1 \in G_1$, $g_1 = a^r$, $r \in \mathbb{Z}$

then $G_1 * G_2 \cong \mathbb{Z} * \mathbb{Z}_2 = \{e, a^p, b, a^p b, ba^p, a^p b a^q, ba^p ba^q, a^p ba^q ba^r b, \ldots \mid p, q, r \in \mathbb{Z}\}$

$$a^p b a^r b a^q \cdot a^{-p} b = a^p b a^r b a^{q-p} b$$

Lemma 2 Let $X = \bigcup_{\alpha \in J} A_\alpha$ s.t. $A_\alpha \underset{\text{open}}{\subset} X$ $\forall \alpha$ and let $p \in \bigcap_{\alpha \in J} A_\alpha$. Then $\exists$ a natural group hom.

$$\Phi : \mathop{*}_{\alpha \in J} \pi_1(A_\alpha, p) \longrightarrow \pi_1(X, p)$$

s.t. Φ sends each reduced word $[\gamma_1] \cdot [\gamma_2] \cdots [\gamma_r] \in \mathop{*}_{\alpha \in J} \pi_1(A_\alpha, p)$

w/ $[\gamma_i] \in \pi_1(A_{\alpha_i}, p)$ to the concatenation $[\gamma_1] * [\gamma_2] * \cdots [\gamma_N] \in \pi_1(X, p)$ and Φ is surjective.

(Already proved this).

If we find $\ker \Phi$ then by the 1st isomorphism thm,

$$\ast \pi_1(A_\alpha, p) \Big/ \ker \Phi \cong \pi_1(X, p).$$

$\boxtimes$

Prop $\{G_\alpha\}_{\alpha \in J}$ is a collection of groups.

1) $\forall \alpha \in J$, $\underset{\alpha \in J}{\ast} G_\alpha$ contains a subgp. isomorphic to G_α.; the subgroup is the empty word union all the reduced words of exactly one letter from G_α.

2) $G_\alpha \leq \underset{\beta \in J}{\ast} G_\beta$, then $\forall \alpha, \gamma \in J$, $\alpha \neq \gamma$, $G_\alpha \cap G_\gamma = \{e\}$ and $g \in G_\alpha$, $h \in G_\gamma$ then $gh \neq hg$ in $\underset{\beta \in J}{\ast} G_\beta$.

3) For any group H w/ hom. $\{\Phi_\alpha : G_\alpha \longrightarrow H\}_{\alpha \in J}$ $\exists!$ hom $\Phi : \underset{\alpha \in J}{\ast} G_\alpha \longrightarrow H$ st $\Phi|_{G_\alpha} = \Phi_\alpha$ $\forall \alpha \in J$.

$g \in \underset{\alpha \in S}{*} G_\alpha \Rightarrow g = g_1 g_2 \cdots g_N \quad , g_i \in G_{\alpha_i}$

$\Rightarrow \Phi(g) = \Phi(g_1 \cdot g_2 \cdots g_N) = \Phi_{\alpha_1}(g_1) \cdot \Phi_{\alpha_2}(g_2) \cdots \Phi_{\alpha_N}(g_N)$

$x, y \in G$, x and y are said to be conjugates iff $x = gyg^{-1}$ for some $g \in G$.

$N \triangleleft G$ normal subgroup iff it is invariant under conjugation by arbitrary elements of G or it contains all of its conjugates or

$$gng^{-1} \in N \quad \forall \ g \in G, n \in N.$$

"$gNg^{-1} = N$"

- Kernel of a hom. is always normal

$\ker \Phi = \{ g \in G \mid \Phi(g) = e \}$

$g \ker \Phi g^{-1} = gkg^{-1}$, $k \in \ker \Phi$.

↳ want to check if $gkg^{-1} \in \ker \Phi$.

$$\Phi(gkg^{-1}) = \Phi(g) \cdot \Phi(k) \cdot \Phi(g^{-1}) = \Phi(g) \cdot \Phi(g)^{-1} = e$$

$\Rightarrow gkg^{-1} \in \ker \Phi \triangleleft G$.

→ $G/N = \{ gN \mid g \in G \}$ is a group $\Longleftrightarrow N \triangleleft G$.

$(aN)(bN) = (ab)N \in G/N$.

Part II (during the prob. session)

$$\Phi : \mathop{*}_{\alpha \in S} \pi_1(A_\alpha, p) \longrightarrow \pi_1(X, p)$$

is determined by the hom. $(j_\alpha)_* : \pi_1(A_\alpha, p) \to \pi_1(X, p)$

$\forall \; j_\alpha : A_\alpha \hookrightarrow X$.

consider

$$j_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha \quad \text{inclusion}$$

$$j_{\beta\alpha} : A_\alpha \cap A_\beta \hookrightarrow A_\beta$$

$$\begin{tikzcd}
 & A_\alpha \arrow[dr, "i_\alpha"] & \\
A_\alpha \cap A_\beta \arrow[ur, "j_{\alpha\beta}"] \arrow[dr, "j_{\beta\alpha}"'] \arrow[rr, phantom, "\circlearrowright"] & & X \\
 & A_\beta \arrow[ur, "i_\beta"'] &
\end{tikzcd}$$

$$i_\alpha \circ j_{\alpha\beta} = i_\beta \circ j_{\beta\alpha}$$

If γ is a loop based at p in $A_\alpha \cap A_\beta$, then

$(j_{\alpha\beta})_* [\gamma] \in \pi_1(A_\alpha, p)$, $(j_{\beta\alpha})_* [\gamma] \in \pi_1(A_\beta, p)$

belong to distinct subgroups in $\mathop{*}_{\delta \in S} \pi_1(A_\delta, p)$. Also

$$(i_\alpha)_* (j_{\alpha\beta})_* [\gamma] = (i_\beta)_* (j_{\beta\alpha})_* [\gamma] \in \pi_1(X, p)$$

$\therefore$

$$\Phi\left((j_{\alpha\beta})_*[\gamma]\right) = \Phi\left((j_{\beta\alpha})_*[\gamma]\right) \Longrightarrow$$

$\mathrm{Ker}(\Phi)$ must contain the reduced word which is formed by the letters $(j_{\alpha\beta})_*[\gamma] \in \pi_1(A_\alpha, p)$ and $(j_{\beta\alpha})_*[\gamma]^{-1} \in \pi_1(A_\beta, p)$, i.e,

$$\underline{(j_{\alpha\beta})_*[\gamma]\left((j_{\beta\alpha})_*[\gamma]\right)^{-1}} \in \mathrm{Ker}\,\Phi$$

$\therefore$ $\mathrm{Ker}\,\Phi$ must contain the smallest normal subgroup of $\underset{\delta\in J}{*}\pi_1(A_\delta, p)$ which contains elements of the underlined form.

Def^n Let G be a group and let S be any subset of G.

$\langle S\rangle \leq G$ is the smallest subgroup of G that contains S $\equiv \bigcap_{\substack{S\subset H\\ H\leq G}} H \equiv \langle S\rangle$ is the set of all products of elements $g\in S$ and $g^{-1}\in S$.

Similarly $\langle S\rangle_N \trianglelefteq G$ is the smallest normal subgroup of G that contains $S \equiv \bigcap_{\substack{S\subseteq N\\ N\trianglelefteq G}} N \equiv \langle S\rangle_N$ is the set of all conjugates of products of elements in S and their inverses.

<u>Theorem</u> (Seifert – van Kampen)

Suppose $X = \bigcup_{\alpha \in J} A_\alpha$, $A_\alpha \underset{\text{open}}{\subset} X$ $\forall \alpha$. w/ non-empty intersection, $i_\alpha : A_\alpha \hookrightarrow X$ and $j_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$, $\forall \alpha, \beta \in J$ and fix $p \in \bigcap_{\alpha \in J} A_\alpha$.

(1) If $A_\alpha \cap A_\beta$ is path-connected for every pair $\alpha, \beta \in J$ then the hom.

$$\Phi : \ast_{\delta \in J} \pi_1(A_\delta, p) \longrightarrow \pi_1(X, p) \text{ is surjective.}$$

(already proved)

(2) If $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected for every triple $\alpha, \beta, \gamma \in J$, then

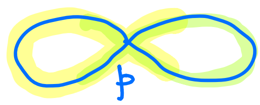
$$\ker \Phi = \underbrace{\left\langle \left\{ (j_{\alpha\beta})_* [\gamma] \left((j_{\beta\alpha})_* [\gamma]^{-1} \right) \;\middle|\; \begin{array}{l} \alpha, \beta \in J \\ [\gamma] \in \pi_1(A_\alpha \cap A_\beta, p) \end{array} \right\} \right\rangle}_{N}$$

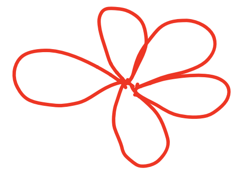
$\therefore$ we have an isomorphism

$$\pi_1(X, p) \cong \ast_{\delta \in J} \pi_1(A_\delta, p) \Big/ \ker \Phi .$$

Remark :- most of the time, X can be covered by two subsets $X = U \cup V$, then we just need that $U \cap V$ is path connected.

Ex. figure 8 or wedge sum of two circles $S^1 \vee S^1$ or bouquet of two circles.

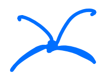




$S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$

V

U

$U \cap V =$  path connected.

$$\pi_1(U, p) \cong \pi_1(S^1) \cong \mathbb{Z}$$

$$\pi_1(V, p) \cong \pi_1(S^1) \cong \mathbb{Z}$$

$U \cap V$ is a contractible $\Rightarrow \pi_1(U \cap V, p) = 0$.

$\therefore$ the Ker Φ in the statement of the van Kampen thm is trivial $\Rightarrow$

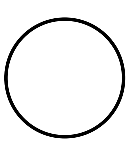
$$\pi_1(S^1 \vee S^1, p) \cong \underset{\langle a \rangle}{\underset{\shortparallel}{\mathbb{Z}}} * \underset{\langle b \rangle}{\underset{\shortparallel}{\mathbb{Z}}} = \text{free group on two generators.}$$

$$\shortparallel$$

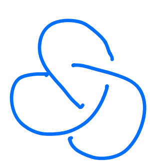
$$\{e, a^p, b^q, a^p b^q, b^p a^q, \dots \mid p, q \in \mathbb{Z}\}$$

non-abelian.

<u>Knots</u> K is a knot then it is just an embedding of S^1 in $\mathbb{R}^3$.



K_0 – unknot



K_1

trefoil knot.

<u>Knot group</u> $\pi_1(\mathbb{R}^3 \setminus K) \rightsquigarrow$ knot group of K.

$\pi_1(\mathbb{R}^3 \setminus K_1)$ and $\pi_1(\mathbb{R}^3 \setminus K_0)$.

<u>Defn</u> Given a set S, the free group on S is defined as

$$F_S = \underset{\alpha \in S}{*} \mathbb{Z}$$

i.e., F_S is the set of all reduced words $a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$, $n \geq 0$, $p_i \in \mathbb{Z}$, $p_i \neq 0$, $a_i \in S$ w/ $a_i \neq a_{i+1}$ $\forall i$.

The elements of S are called generators of F_S.

<u>Lemma</u> Every group is isomorphic to a quotient of a free group.

<u>Proof.</u> Let G be a group. Pick any subset $S \subset G$ s.t. $\langle S \rangle = G$. Then the hom. $\Phi : F_S \longrightarrow G$

$\Phi(g) \longmapsto g$ is surjective $\Rightarrow$ by the 1st iso. thm
$\underset{S \subset F_S}{\cap}$

$$G \cong F_S / \ker \Phi . \qquad \boxtimes$$

<u>Defn</u> Given a set S, a relation in S means any equation of the form "$a = b$" where $a, b \in F_S$.

<u>Defn</u> For any set S and a set R consisting of relations in S, we define the group

$$\{S \mid R\} = F_S / \langle R' \rangle_N$$

where R' is the set of all elements of the form $ab^{-1} \in F_S$ for relations "$a = b$" in R.

The elements of S are called the generators of this group and the elements of R are its relation.

Any element $\in \{S \mid R\}$ is a reduced word $w \rightsquigarrow [w]$ using letters in S

$w \neq w' \in F_S$ but it can happen that $[w] = [w']$.

This will happen $\iff w^{-1}w' \in \langle R' \rangle_N$

$\iff$ "$w = w'$" is one of the relations.

Every group is isomorphic to $\{S \mid R\}$ for some set of generators S and relations R.

(in the previous lemma, take S to be the set of generators itself and define R to consist of all relations "$a = b$" s.t. $ab^{-1} \in \ker \Phi$).

Defn G is a group, choice of generators is S and relations in S. $G \cong \{S \mid R\}$ is called a presentation of G.

We say that G is finitely presented if it admits a presentation s.t. S and R are both finite sets.

Corr. (Seifert – van Kampen thm. for finitely presented groups)

$X = A \cup B$, $A, B \underset{\text{open}}{\subset} X$, path-connected.

$A \cap B$ is path-connected.

$j_A : A \cap B \hookrightarrow A$, $j_B : A \cap B \hookrightarrow B$

Suppose we have finite presentations,

$\pi_1(A) \cong \langle \{a_i\} \mid \{R_j\} \rangle$

$\pi_1(B) \cong \langle \{b_k\} \mid \{S_\ell\} \rangle$ $\quad$ i, j, k, ℓ, p, q take only finitely many values.

$\pi_1(A \cap B) \cong \langle \{c_p\} \mid \{T_q\} \rangle$

Then

$$\pi_1(X) \cong \langle \{a_i\} \cup \{b_k\} \mid \{R_j\} \cup \{S_\ell\} \cup \{(j_A)_* c_p = (j_B)_* c_p\} \rangle$$

$\blacksquare$

<u>Ex</u>:- 1) $\langle a \rangle = \langle a \mid \phi \rangle$ is isomorphic to $F_{\{a\}} \cong \mathbb{Z}$

2) $\langle a \mid a^p = e \rangle \cong \mathbb{Z}_p \equiv \mathbb{Z}/p\mathbb{Z}$

3) $\langle a, b \mid ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}$.