Lecture 17

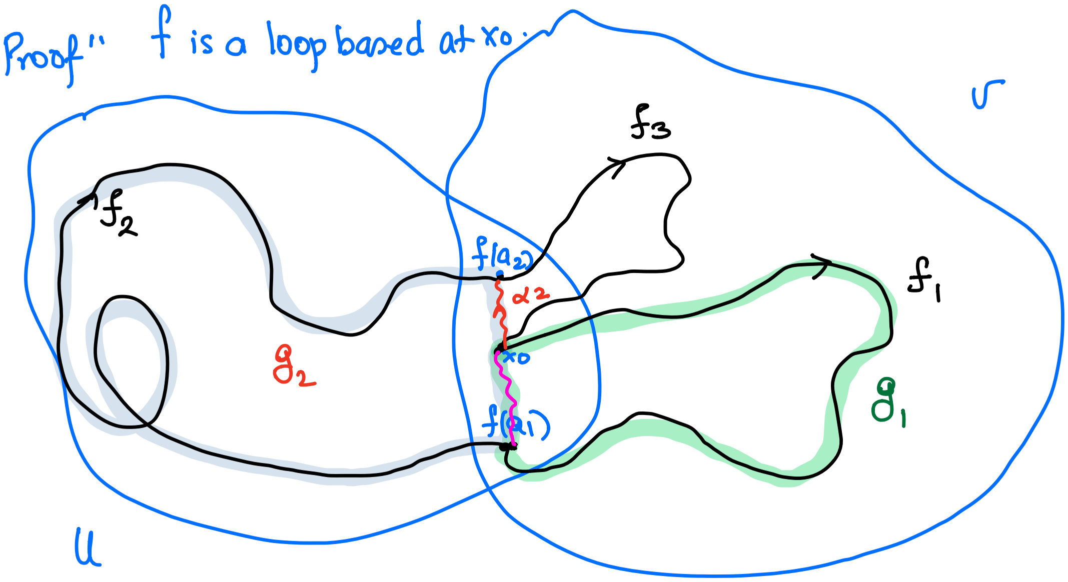
Recall

Theorem Suppose $X = U \cup V$, $U, V \underset{\text{open}}{\subseteq} X$. Suppose $x_0 \in U \cap V$ and $U \cap V$ is path-connected. Look at $i : U \hookrightarrow X$, $j : V \hookrightarrow X$. The images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0), \quad j_* : \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

"Proof" f is a loop based at x_0.



subdivision $a_0 < a_1 < \cdots < a_m$ of $[0,1]$ s.t.
$f(a_i) \in U \cap V$ $\forall i$ and $f([a_{i-1}, a_i]) \subset U$ or V.

$$\underbrace{[0,1] \xrightarrow{\text{plm}} [a_{i-1}, a_i] \xrightarrow{f} X}_{f_i} \qquad [f] = [f_1] * [f_2] * \cdots * [f_n]$$

$$g_i = (\alpha_{i-1} * f_i) * \alpha_i^{-1}$$

$\forall i$, α_i is a path in $U \cap V$ from x_0 to $f(a_i)$

α_0 and α_n is just the constant path at x_0.

$$[g_1] * [g_2] * \cdots * [g_n] = [f_1] * [f_2] * \cdots * [f_n] = [f].$$

$\boxtimes$

$$\pi_1(S^n, b_0) = \{0\} \quad \forall\ n \geq 2$$

$$U = S^n \setminus \{p\} \cong \mathbb{R}^n, \quad V = S^n \setminus \{q\} \cong \mathbb{R}^n$$

$$\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$$

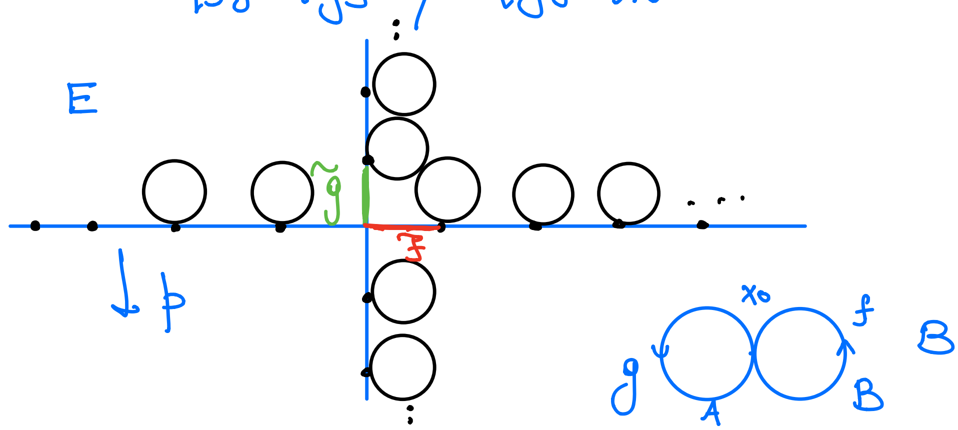
S^n is a universal 2-fold cover of $\mathbb{RP}^n$.

<u>**Lemma**</u> The fundamental group of figure eight is not abelian.

<u>Proof.</u> Idea:- - Get a covering space of fig. 8

- Take two elements $[f], [g] \in \pi_1(8, x_0)$
- Use the path lifting lemma to show that $f * g$ and $g * f$ are not path homotopic

$\Longrightarrow [f] * [g] \neq [g] * [f]$.



For the covering map, wrap the x-axis around the circle A, wrap the y-axis around B.

Each circle tangent to the integer points on the x-axis is mapped homeomorphical onto circle B.

" ———————————————— " ——— on the y-axis

" ———————————— " circle A.

All the integer points are mapped to x_0.

Consider $\tilde{f} : I \longrightarrow E$ $\quad \tilde{f}(s) = (s,0)$ from $(0,0)$ to $(1,0)$

$\tilde{g} : I \longrightarrow E$ $\quad \tilde{g}(s) = (0,s)$ from $(0,0)$ to $(0,1)$

Let $f = p \circ \tilde{f}$ loop based at x_0 in fig. 8

$g = p \circ \tilde{g}$ " ———— " x_0 in fig. 8.

$f * g$ and $g * f$ are loops at x_0.

Claim :- $f * g$ and $g * f$ are not path-homotopic.

If they were path-homotopic then from the section on the path-lifting property, we know the ends points of $\widetilde{f * g}$ and $\widetilde{g * f}$ must be the same.

But the way we have described p, one path ends at $(1,0)$ and the other at $(0,1)$

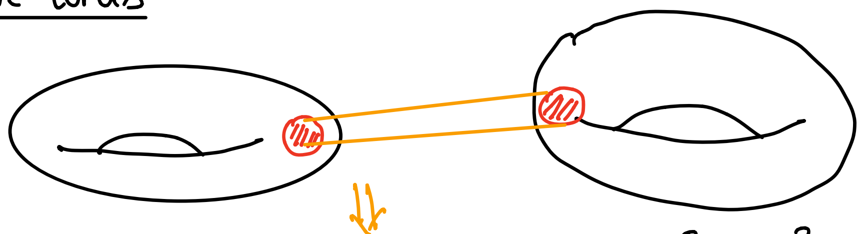
$$\Longrightarrow \quad f * g \not\simeq_p g * f \Longrightarrow [f] * [g] \neq [g] * [f]$$

$\Rightarrow \quad \pi_1(8, x_0)$ is non-abelian. $\boxed{//}$

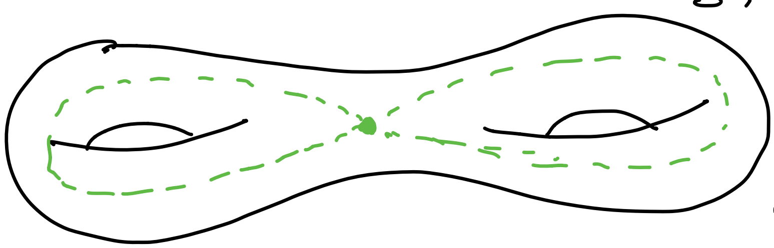
Remark Later we'll see that $\pi_1(8, x_0)$ is a free group on two generators.

$\pi_1(\mathbb{T}^2 \setminus \{p\}, x_0)$ is also non-abelian and is a free group on two generators.

Double Torus



$S^2, \mathbb{RP}^2, \mathbb{T}^2, \mathbb{T} \# \mathbb{T}$

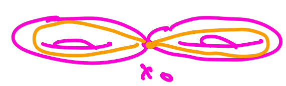


double torus

$\mathbb{T} \# \mathbb{T}$

surface of genus 2.

Check :- $\mathbb{T} \# \mathbb{T}$ retracts to fig. 8.



$\Rightarrow \pi_1(\mathbb{T} \# \mathbb{T})$ ~~is isomorphic to free group on two generators~~. has a subgroup that is isomorphic to free group of two generators.

Theorem The 2-sphere S^2, $\mathbb{RP}^2$, $\mathbb{T}$, $\mathbb{T} \# \mathbb{T}$ are topologically distinct.

$\pi_1(S^2) = \{0\}$ $\qquad$ $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$

$\pi_1(\mathbb{T}) \cong \mathbb{Z} \times \mathbb{Z}$ $\qquad$ $\pi_1(\mathbb{T} \# \mathbb{T}) \cong$ free gp on 2 generators.

Digression into Group theory

$\langle a \rangle = \{ a^n \mid n \in \mathbb{Z} \}$ — cyclic group generated by a.

$a^0 = e$, $a^n = \underbrace{a \cdot a \cdots a}_{n\text{-times}}$

$a^n \cdot a^m = a^{n+m}$

$(a^{-m}) = (a^{-1})^m$

$(G, \cdot)$ -group.

$\varphi : G \to H$ homomorphism if

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$$

$\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$

$2 = 1+1$

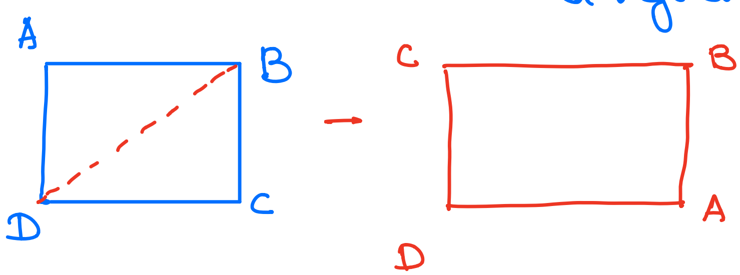
$\mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_5, \ldots$

$H \le G$ (subgroup), if $H \subseteq G$ and is a group with the same operation as in G.

$$\cancel{(\mathbb{Z}_3 \supseteq \mathbb{Z}_2)}$$

__Normal Subgroup__ A subgroup $N \le G$ is a normal subgroup if $\forall g \in G$, $gNg^{-1} \subseteq N$, i.e. $gng^{-1} \in N$ $\forall n \in N$.

- any subgroup of an abelian group is a normal subgroup.
- D_{2n} - dihedral group - group of symmetries of a regular n-gon.



subgroup of rotations is a normal subgroup of D_{2n}.

- S_n - group of permutations on n letters

$\{ f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}, \text{ bijections} \}$

A_n - alternating groups. , normal subgroup of S_n.

$N \trianglelefteq G \rightsquigarrow$ N is a normal subgroup of G.

<u>Cosets</u>:- $H \leq G$, $g_1 H = \{ g_1 h \mid h \in H \}$ coset of H in G

$$\frac{G}{H} = \{ g_1 H, g_2 H, g_3 H, \dots \}$$

G/H will be a group $\Longleftrightarrow$ $H \trianglelefteq G$.

$$(g_1 H) \cdot (g_2 H) = (g_1 g_2) H$$

First Isomorphism Theorem

Let $\varphi : G \longrightarrow H$ be a homomorphism. Denote by $\ker\varphi$, the kernel of the hom. φ.

$$\ker\varphi = \{ g \in G \mid \varphi(g) = e_H \}.$$

$\ker\varphi \trianglelefteq G$.

$$G/\ker\varphi \cong \operatorname{im}(\varphi) \leq H.$$

$\therefore$ if φ is surjective then $G/\ker\varphi \cong H$.

Direct Sum of abelian groups

G is an abelian and let $\{G_\alpha\}_{\alpha \in J}$ is an indexed family of subgroups of G.

We say that G_α generate G if every element $g \in G$ can be written as a finite sum of elements of G_α.

viewing the group operation as addition.

$$x = x_{\alpha_1} + x_{\alpha_2} + \cdots + x_{\alpha_n} \quad \text{s.t.} \quad \alpha_i \neq \alpha_j .$$

$x = \sum_{\alpha \in J} x_\alpha$ then $x_\alpha = 0$ for all but finitely many α's.

x_α = identity element