Lecture 16

* no problem set this week. Next problem session will cover some other topics from the course.

Recall:-

Deformation retract :- $A \subset X$, deformation retract
$\exists\ H : X \times I \to X$ s.t. $H(x,0) = x$ and $H(x,1) \in A$
and $H(a,t) = a \quad \forall\ a \in A,\ t \in I$.

$r(x) = H(x,1)$ retraction of X onto A.

$j : A \hookrightarrow X$ induces an isomorphism of fundamental groups.

Homotopy type

$f : X \to Y,\ g : Y \to X$ if
$f \circ g : Y \to Y \simeq id_Y$ and $g \circ f : X \to X \simeq id_X$

f and g are homotopy equivalence.

Lemma Let h and $k : X \to Y$ be cont. maps and $h(x_0) = y_0$ and $k(x_0) = y_1$. If $h \simeq k$ then $\exists$ a path α in Y from y_0 to y_1 s.t. $k_* = \hat{\alpha} \circ h_*$.

$\hat{\alpha}([f]) = [\alpha]^{-1} * [f] * [\alpha]$.

If $H : X \times I \to Y$ is the hom. b/w h and k then
$\alpha(t) = H(x_0, t)$.

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y, y_1)
\end{tikzcd}$$

<u>Proof</u> :- $f : I \to X$ is a loop at x_0

Want:- $k_*([f]) = \hat{\alpha}(h_*([f]))$

$\implies [k \circ f] = [\alpha]^{-1} * [h \circ f] * [\alpha]$

$\implies [\alpha] * [k \circ f] = [h \circ f] * [\alpha]$ —— (1)

Consider loops f_0 and f_1 in $X \times I$

$f_0(s) = (f(s), 0)$ and $f_1(s) = (f(s), 1)$

consider the path c in $X \times I$, given by
$$c(t) = (x_0, t)$$

Note, $H \circ f_0 = h \circ f$, $H \circ f_1 = k \circ f$, $H \circ c = \alpha$

Suppose $F : I \times I \to X \times I$ be the map
$F(s,t) = (f(s), t)$ Then, the paths in $I \times I$

$\beta_0(s) = (s,0)$ and $\beta_1(s) = (s,1)$

$\gamma_0(t) = (0,t)$ and $\gamma_1(t) = (1,t)$

Then $F\circ\beta_0 = f_0$, $F\circ\beta_1 = f_1$, $F\circ\gamma_0 = F\circ\gamma_1 = c$

$\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I\times I$ from $(0,0)$ to $(1,1)$, $\because$ $I\times I$ is convex $\Rightarrow$ $\underbrace{\beta_0 * \gamma_1 \simeq_p \gamma_0 * \beta_1}_{G}$

$F\circ G$ is a path hom. in $X\times I$ b/w $f_0 * c$ and $c * f_1$

and $H\circ(F\circ G)$ is a path hom. in Y b/w

$$(H\circ f_0) * (H\circ c) = (h\circ f) * \alpha$$ and

$$(H\circ c) * (H\circ f_1) = \alpha * (k\circ f)$$

which proves 1 and hence the lemma. $\boxtimes$

Cor: $h, k : X \to Y$ homotopic, cont. maps w/ $h(x_0) = y_0$ $k(x_0) = y_1$. Then if h_* is injective, surjective or trivial then so is k_*.

$$k_* = \hat{\alpha}\circ h_*$$

Cor. $h: X \to Y$ is nullhomotopic then h_* is trivial.

Theorem:- Let $f: X \to Y$ be cont. w/ $f(x_0) = y_0$. If f is a homotopy equivalence (or X and Y have the same homotopy type), then

$$f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism. (spaces having the same hom. type have isomorphic fundamental groups)

Proof Let $g: Y \longrightarrow X$ hom. inverse of f.

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$$

$$x_1 = g(y_0), \quad y_1 = f(x_1)$$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(f_{x_0})_*"] & \pi_1(Y, y_0) \arrow[d, "g_*"] & \\ & \pi_1(X, x_1) \arrow[r, "(f_{x_1})_*"] & \pi_1(Y, y_1) \end{tikzcd}$$

$g \circ f: (X, x_0) \longrightarrow (X, x_1) \simeq id_X \implies$
$\exists$ path α in X from x_0 to x_1 s.t

$$(g \circ f)_* = \hat{\alpha} \circ (id_X)_* = \hat{\alpha}$$

$\Rightarrow$ $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism.

Similarly, for $(f \circ g) \simeq id_Y$ $\Rightarrow$

$(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism.

g_* is surjective $\quad$ g_* is injective.

$\Rightarrow$ g_* is an isomorphism

and $\quad (f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$

$\Rightarrow$ $(f_{x_0})_*$ is an isomorphism

$\blacksquare$

Remark :- Even though g is the homotopy inverse of f, the induced isomorphisms are NOT the inverses of each other.

Seifert - von Kampen Theorem

Theorem Suppose $X = U \cup V$ where U and V are open sets of X. Suppose that $U \cap V$ is path connected and $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the inclusion mappings. Then the images of the induced homomorphisms

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and}$$

$$j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

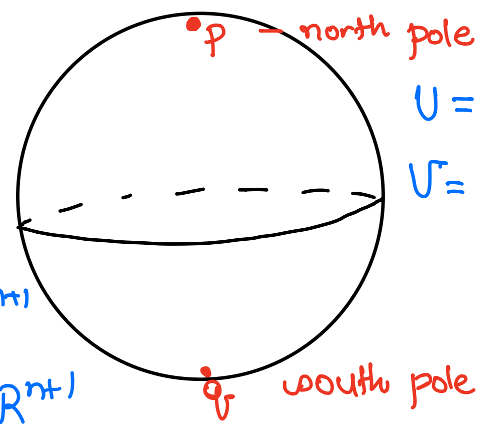
generate $\pi_1(X, x_0)$, i.e., suppose f is a loop in X based at x_0 then f is path homotopic to a product of the form $(g_1 * (g_2 * (g_3 * (\cdots * g_n))))$

where each g_i is a loop in X based at x_0 and it lies either in U or in V.

<u>Corollary</u>:- $X = U \cup V$, U, V open sets in X, $U \cap V \neq \phi$, path connected. If U and V are simply connected then X is simply connected.

<u>Theorem</u> If $n \geq 2$, then n-sphere S^n is simply connected.

<u>Proof</u> :-



$U = S^n - \{p\}$

$V = S^n - \{q\}$

$p = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$

$q = (0, 0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$

For $n \geq 1$, the punctured sphere $S^n - p$ is homeomorphic to $\mathbb{R}^n$. (U and V are simply conneted).

$f : S^n - p \longrightarrow \mathbb{R}^n$ stereographic projection.

$$f(x) = f(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$$

$f^{-1} = g : \mathbb{R}^n \longrightarrow (S^n - p)$ given by

$$g(y) = g(y_1, \dots, y_n) = \left(\underbrace{t(y)}_{t(y) = \frac{2}{(1 + \|y\|^2)}} \cdot y_1, \dots, t(y) \cdot y_n, \ 1 - t(y) \right)$$

$U = S^n - p$, $V = S^n - q$ open sets in the corr.

$U \cap V$ is path connected.

$U \cong \mathbb{R}^n$ and $V \cong \mathbb{R}^n \Rightarrow$ simply connected

$\Rightarrow \quad \pi_1 (S^n, b_0) = \{0\}$, $n \geq 2$.

Fundamental Group of $\mathbb{R}P^2$ ($\mathbb{R}P^n$, $n \geq 2$).

$\mathbb{R}P^n = S^n / \sim$, $x \sim -x$, $x \in S^n$.

$p : S^n \longrightarrow \mathbb{R}P^n$ the quotient map.

Cor: $\pi_1(\mathbb{RP}^2, y)$ is a group of order $2 \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

$\pi_1(\mathbb{RP}^n, y) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, $n \geq 2$.

proof We show $p: S^2 \longrightarrow \mathbb{RP}^2$ is a covering map. Granted this, we know $\pi_1(\mathbb{RP}^2, y)$ as a set is in bijective correspondence $p^{-1}(y)$.

$\downarrow$ has exactly 2 elements $y, -y$.

$\Downarrow$

$\pi_1(\mathbb{RP}^2, y) \cong \mathbb{Z}_2$.

Exer:- Show that $p: S^2 \to \mathbb{RP}^2$ is a covering map, p is a 2-fold covering map. $\boxtimes$

Proof of the theorem

Step 1 $\exists$ a subdivision $a_0 < a_1 < \cdots < a_n$ of $[0,1]$ s.t. $f(a_i) \in U \cap V$ and $f([a_{i-1}, a_i]) \subset U$ or V $\forall i$.

f is a loop based at x_0.

Choose a subdivision $b_0, b_1, \ldots b_m$ of $[0,1]$ s.t. $\forall i$, $f([b_{i-1}, b_i]) \subset U$ or V. If $f(b_i) \in U \cap V$

$\forall i$ then we are done.

If not, then let i be an index st. $f(b_i) \notin U \cap V$.

Then $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}]) \subset U$ or V.

If $f(b_i) \in U$ then must lie in U

if $f(b_i) \in V$ then " ———— " in V

Delete this point b_i from the subdivision and obtain a smaller subdivision $c_0, c_1, \dots, c_{m-1}$ that satisfies $f([c_{i-1}, c_i]) \subset U$ or V $\forall i$.

Repeat this procedure finitely many times gives the required subdivision.

<u>Step 2</u> Given f, $a_0, a_1, \dots, a_n$ be the subdivision in Step 1.

Define $f_i = f * \text{plm of } [0,1] \to [a_{i-1}, a_i]$

$[a,b] \longrightarrow [c,d]$ plm $y = mx + k$

st $a \mapsto c$
$b \mapsto d$.

The path f_i lies either in U or in V

$$[f] = [f_1] * [f_2] * \cdots * [f_n]$$

$\forall i$, choose a path α_i from x_0 to $f(a_i)$ in

$U \cap V$.

$\because$ $f(a_0) = f(a_n) = x_0 \Rightarrow$ let's choose α_0 and α_n to be the constant path at x_0.

Set $$g_i = (\alpha_{i-1} * f_i) * \alpha_i^{-1} \quad \forall i$$

$\perp$ loop in X based at x_0, its image lies either in U or in V.

$$g_i * g_{i+1} = \left((\alpha_{i-1} * f_i) * \alpha_i^{-1} \right) * \left(\alpha_i * f_{i+1} \right) * \alpha_{(i+1)}^{-1}$$

$$= \alpha_{i-1} * f_i * f_{i+1} * \alpha_{(i+1)}^{-1}$$

$$[g_1] * [g_2] * \cdots * [g_n] = [f_1] * [f_2] * \cdots * [f_n]$$

$\boxed{//}$