Lecture 15

Recall :- • r is a retraction of X onto A then $j : A \hookrightarrow X$ induces an injective hom.

• Brouwer's fixed pt. thm :- $f : B^2 \longrightarrow B^2$ continuous then $\exists\ x \in B^2$ s.t. $f(x) = x$.

• Borsuk-Ulam thm.

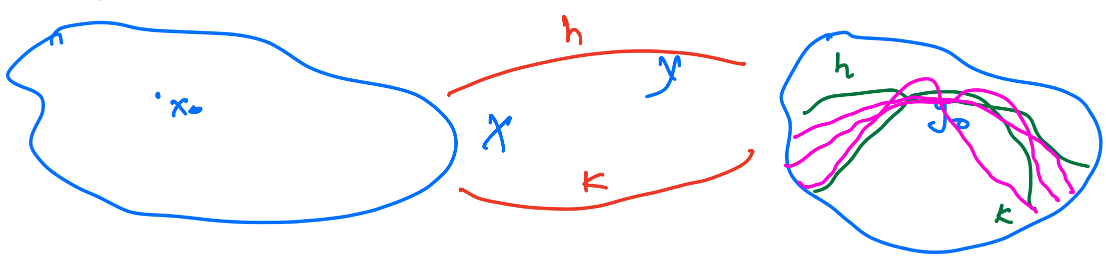
• Topological proof of FTA.

For calculating $\pi_1(B, b_0)$, try to study its covering spaces.

Homotopy type of a space.

$\Big\}$ help us in computing the fundamental gp. of a space using some other $\pi_1(X)$ where X is familiar/easier space.

Lemma Let $h, k : (X, x_0) \longrightarrow (Y, y_0)$ be cont. maps. If h and k are homotopic, and if the image of the base point $x_0 \in X$ remains fixed at $y_0 \in Y$ during the homotopy then $h_* = k_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$.



Proof: $\exists$ homotopy $H: X \times I \longrightarrow Y$ s.t.

$H(x,0) = h$, $H(x,1) = k$ and $H(x_0,t) = y_0$ $\forall t \in I$.

If f is a loop at $x_0 \in X$ then

$$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$$

is a homotopy b/w $h \circ f$ and $k \circ f$, infact, it's a path hom. as f is a loop at x_0 and H maps $x_0 \times I$ to y_0.

$\Rightarrow$ $[h \circ f] = [k \circ f]$

$\Rightarrow$ $h_*([f]) = k_*([f]) \Rightarrow h_* = k_*$. ☒

$\mathbb{R}^2 - \{0\}$ retracts onto $S^1 \rightsquigarrow j: S^1 \hookrightarrow \mathbb{R}^2 - \{0\}$ induces an injective hom. In fact, j_* is an isomorphism b/w $\pi_1(S^1)$ and $\pi_1(\mathbb{R}^2 - \{0\})$.

Theorem The inclusion map $j: S^n \longrightarrow \mathbb{R}^{n+1} - \{0\}$ induces an isomorphism of fundamental groups.

Proof:- $b_0 = (1, 0, 0, \dots, 0)$, let

$r: \mathbb{R}^{n+1} - \{0\} \longrightarrow S^n$, $r(x) = \frac{x}{\|x\|}$, $r|_{S^n} = id_{S^n}$

Then $r \circ j: S^n \longrightarrow S^n$ is the identity map $\Rightarrow$

$(r \circ j)_* = r_* \circ j_*$ is the identity hom. of $\pi_1(S^n, b_0)$.

Consider $j \circ r : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} - \{0\}$

$j \circ r$ is not the identity map, but is homotopic to $id_{\mathbb{R}^{n+1} - \{0\}}$.

$$H : \mathbb{R}^{n+1} - \{0\} \times I \longrightarrow \mathbb{R}^{n+1} - \{0\}$$

$$H(x,t) = (1-t)x + \frac{tx}{\|x\|}, \quad H(x,0) = id_{\mathbb{R}^{n+1} - \{0\}}$$
$$H(x,1) = j \circ r.$$

The point $b_0 = (1, 0, \ldots, 0)$ remains fixed during the homotopy, as $\|b_0\| = 1 \Rightarrow$ by the previous lemma the $(j \circ r)_* = j_* \circ r_* = id_* : \pi_1(\mathbb{R}^{n+1} - \{0\}) \longrightarrow \pi_1(\mathbb{R}^{n+1} - \{0\})$

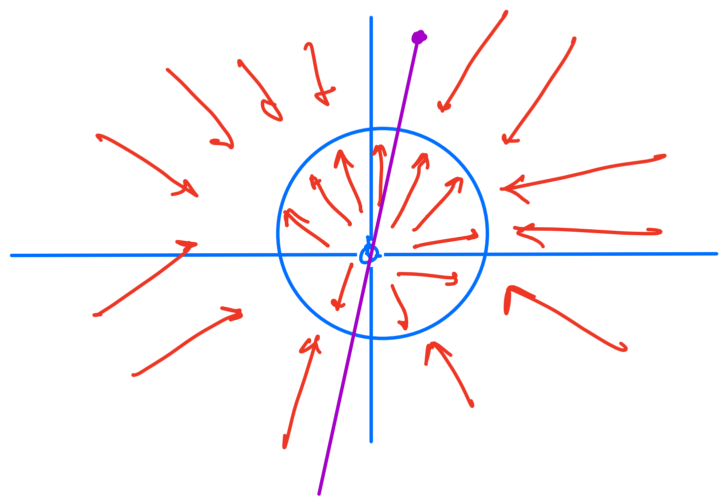
$\therefore$ $j_* : \pi_1(S^n, b_0) \longrightarrow \pi_1(\mathbb{R}^{n+1} - \{0\})$ is an isomorphism.

$\boxed{//}$

straight line homotopy

→ main ideas:- we have a natural way of deforming the $id_{\mathbb{R}^{n+1} - \{0\}}$ to a map $(j \circ r)$ which is collapsing $\mathbb{R}^{n+1} - \{0\}$ onto S^n.

was keeping the subspace (S^n) fixed $\forall t$.



Cor. $\pi_1(S^n) \cong \pi_1(\mathbb{R}^{n+1} - \{0\})$, $\pi_1(\mathbb{R}^2 - \{0\}) \cong (\mathbb{Z}, +)$.

Cor. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ $\forall\, n > 2$.

$\pi_1(\mathbb{R}^2 - \{0\})$ must be iso. to $\pi_1(\mathbb{R}^n - \{0\})$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong (\mathbb{Z}, +)$, $\quad \pi_1(\mathbb{R}^n - \{0\}) \cong \pi_1(S^n) = \{0\}$ for $n \geq 2$.

Defn:- $A \subset X$. We say that A is deformation retract of X if the $id_X \simeq h$ s.t. h carries X to A and each point of A remains fixed during the homotopy, i.e. $\exists$ a cont. map $H : X \times I \to X$ s.t. $H(x, 0) = id_X$, $H(x, 1) \in A$ $\forall\, x \in X$ and $H(a, t) = a$ $\forall\, t \in I$, $\forall\, a \in A$.

The homotopy H is called a deformation retraction of X onto A. $r: X \to A$, $r(x) = H(x,1)$ is a retraction of X onto A., H is a homotopy b/w id_X and the map $j \circ r : X \to X$, $j : A \hookrightarrow X$.

Just like the preceding thm,

Thm. Let A be a deformation retract of X, $a_0 \in A$. Then $j : (A, a_0) \longrightarrow (X, a_0)$ inclusion map, induces an isomorphism of fundamental group. i.e. if A is a deformation retract of X, then

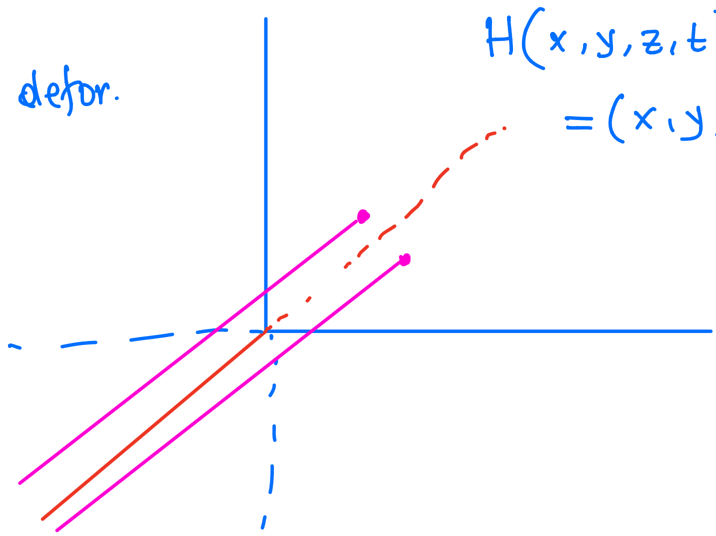
$$\pi_1(A, a_0) \cong \pi_1(X, a_0).$$

$\boxed{//}$

Ex. 1. B = z-axis in $\mathbb{R}^3$ and we look at $\mathbb{R}^3 - B$

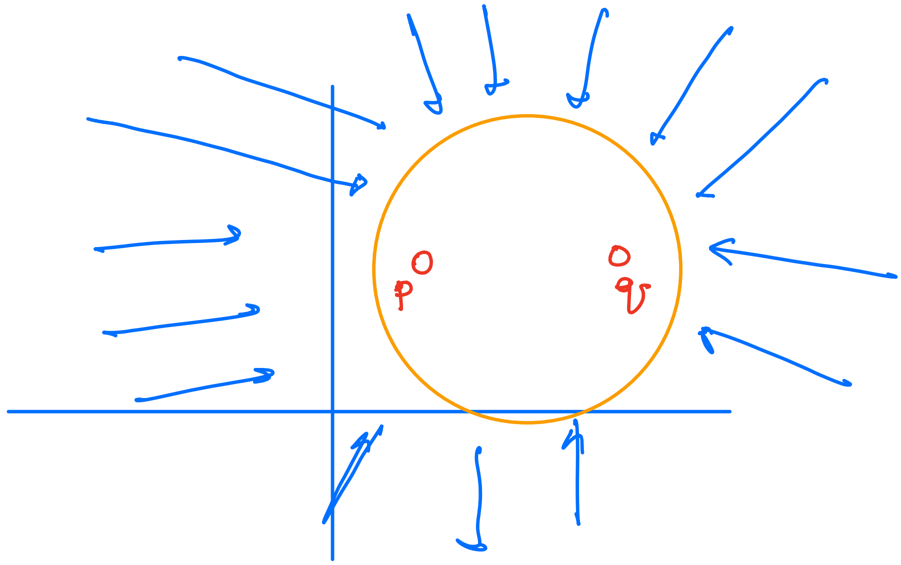
$H(x, y, z, t) = (x, y, (1-t)z)$

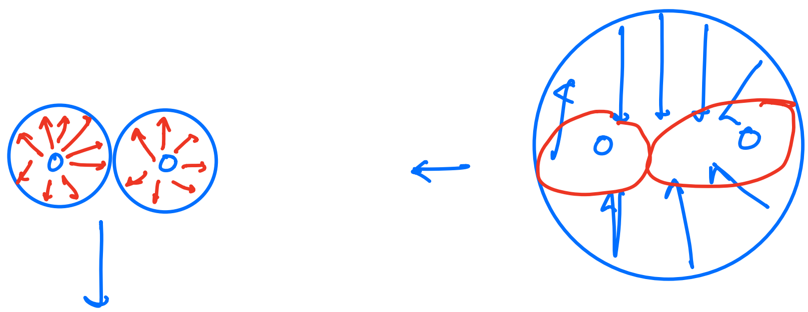
$\mathbb{R}^3 - B$ has a defor. retract which $(\mathbb{R}^2 - \{0\}) \times 0$.

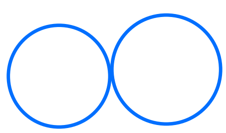


$$\pi_1(\mathbb{R}^3 - B) \cong \pi_1((\mathbb{R}^2 - \{0\}) \times \{0\}) \cong \pi_1(\mathbb{R}^2 - \{0\}) \cong (\mathbb{Z}, +).$$

Ex. Doubly punctured plane $\mathbb{R}^2 - p - q$.







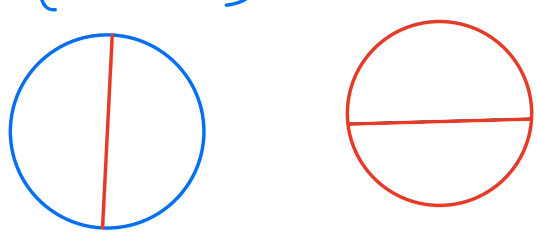
"figure eight" space.

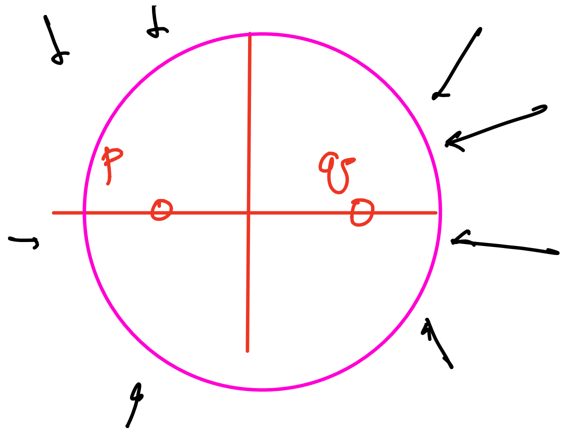
$$\pi_1(\mathbb{R}^2 - p - q) \cong \pi_1(\text{figure eight}) \to \text{non-abelian.}$$

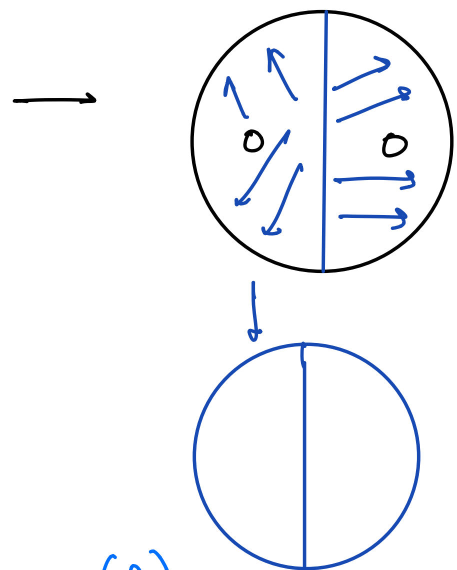
Ex 3

$\mathbb{R}^2 - p - q$ deformation retracts to "theta space"

$$\Theta = S^1 \cup (0 \times [-1,1])$$







$\pi_1(\mathbb{R}^2 - p - q) \cong \pi_1(8) \cong \pi_1(\theta)$.

Def^n Let $f: X \to Y$ and $g: Y \to X$ cont. maps.

Suppose $g \circ f : X \to X \simeq id_X$

$f \circ g : Y \to Y \simeq id_Y$

Then f and g are called homotopy equivalences and f is a homotopy inverse of g and vice-versa.

if $f: X \to Y$ is a hom. equivalence of X w/ Y

$h: Y \to Z$ " ———————— " Y w/ Z

$h \circ f: X \to Z$ " ———————— " X w/ Z.

This relation on topological spaces $X \sim Y$ if X and Y are homotopy equivalent is an equivalence relation.

$[X] = \{ Y \text{ top space} \mid X \text{ is top. equivalent to } Y \}$

Two spaces that are homotopy equivalent are said to have the same homotopy type.

→ If A is def. retract of X then A has the same homotopy type as X.

$j : A \hookrightarrow X$ inclusion

$r : X \to A$ retraction

$r \circ j = id_A$ $\qquad j \circ r \simeq id_X$, $r \circ j$ and $j \circ r$ are homotopy inverses.

<u>Lemma</u> :– Let $h, k : X \to Y$ be cont. maps. Let $h(x_0) = y_0$, $k(x_0) = y_1$. If h and k are homotopic then $\exists$ a path α in Y from y_0 to y_1 s.t. $k_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$ then $k_* = \hat{\alpha} \circ h_*$

$\left(\hat{\alpha}([f]) = [\alpha]^{-1} * [f] * [\alpha] \right)$. In fact, if

$H : X \times I \to Y$ is the homotopy b/w h and k,

then α is the path, $\alpha(t) = H(x_0, t)$.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & {} \end{tikzcd}$$

$k_* \longrightarrow \pi_1(Y, y_1)$

Corr:- If $h, k: X \to Y$ homotopic cont. maps. $h(x_0) = y_0$, $k(x_0) = y_1$. If h_* is injective, bijective, trivial or surjective, then so is k_*.

Corr. Let $h: X \to Y$. If h is nullhomotopic then h_* is the trivial hom.

Proof: The constant map induces the trivial hom. and $h \simeq$ constant map. $\Rightarrow h_*$ is the trivial hom.

Theorem:- Let $f: X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence then

$$f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

is an isomorphism. Spaces with the same homotopy type have isomorphic fundamental groups.