Lecture 14

* PSet 5 will be uploaded after the problem-session.

Recall :-

Theorem:- Let $h: S^1 \longrightarrow X$ be a continuous map. Then TFAE:-

1) h is nullhomotopic.
2) h extends to a continuous map $k: B^2 \to X$.
3) h_* is the trivial hom. of fundamental groups.

i.e., $h_*([f]) = [e]$ in $\pi_1(X, x_0)$.

1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3) done.

(3) $\Rightarrow$ (1) Let $p: \mathbb{R} \to S^1$ be the usual covering map and let $p_0: I \longrightarrow S^1$ be $p|_I$.

Then as we discussed, $[p_0]$ generates the cyclic group $\pi_1(S^1, b_0)$ as $\tilde{p}_0$ starts at 0 and ends at 1.

Let $x_0 = h(b_0)$.

$\because$ $h_*: \pi_1(S^1, b_0) \longrightarrow \pi_1(X, x_0)$ is trivial

$\Rightarrow$ the loop $[f] = [h \circ p_0]$ is the identity element of $\pi_1(X, x_0)$.

$\therefore$ $\exists$ a path hom. in X, F b/w f and e_{x_0}.

We note that $p_0 \times id : I \times I \longrightarrow S^1 \times I$ is a quotient map which is injective apart from

$$\left.\begin{matrix} 0 \times t \\ 1 \times t \end{matrix}\right\} \longrightarrow b_0 \times t \qquad \forall\ t \in I.$$

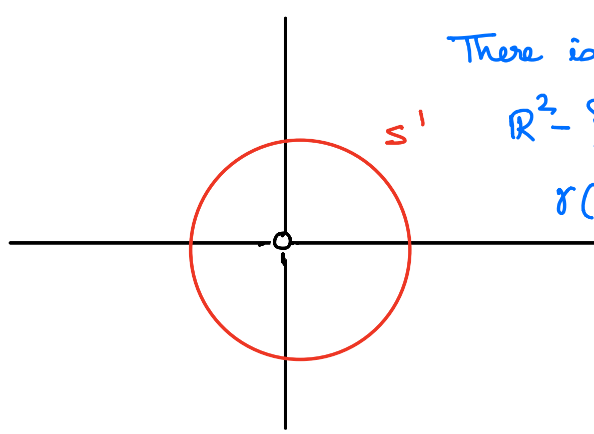
moreover $F(0 \times I) = F(1 \times I) = F(I \times 1) = x_0 \in X$

$\therefore$ from the theorem on continuous maps of quotient spaces $\exists$ a continuous map $H : S^1 \times I \longrightarrow X$ which is a homotopy b/w h and a constant map.
$\Rightarrow$ h is nullhomotopic. $\blacksquare$

$\therefore$

Corr.:- The inclusion map $j : S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ is not nullhomotopic. The $id : S^1 \longrightarrow S^1$ is not nullhomotopic.

Proof:-



There is a retraction of $\mathbb{R}^2 - \{0\}$ onto S^1.

$$r(x) = \frac{x}{\|x\|}$$

$\Rightarrow$ j_* is injective $\Rightarrow$ j_* cannot be trivial

$\therefore$ by the previous thm, j is not nullhomotopic.

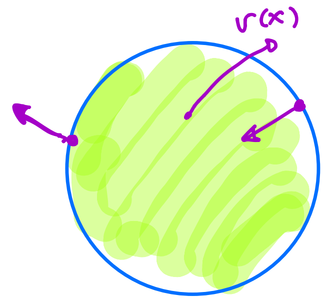
$id: S^1 \to S^1 \implies id_*: \pi_1(S^1, b_0) \to \pi_1(S^1, 0)$

$\neq$ trivial hom. $\implies$ id. is also not nullhomotopic.

$\blacksquare$

If $B^2 \subseteq \mathbb{R}^2$, then a vector field on B^2 is an ordered pair $(x, v(x))$, $x \in B^2$ and v is a continuous map from $B^2 \to \mathbb{R}^2$.

<u>Theorem</u> :- Given a nonvanishing vector field on B^2, $\exists$ a point of S^1 where the vector field points directly inward and a point of S^1 where the v.f. points directly outwards.



<u>Proof</u> :- Let $v(x)$ be the v.f. on B^2. nonvanishing means

$v(x) \neq 0 \in \mathbb{R}^2 \quad \forall\, x \in B^2.$

$v: B^2 \to \mathbb{R}^2 - \{0\}.$

Suppose $v(x)$ doesn't point inward at any point $x \in S^1$

$v: B^2 \to \mathbb{R}^2 - \{0\}$

$w = v|_{S^1} : S^1 \to \mathbb{R}^2 - \{0\} \implies$ from the equivalence of 1) and 2) in the previous thm, $w: S^1 \to \mathbb{R}^2 - \{0\}$

is nullhomotopic.

If we can produce a homotopy b/w w and the inclusion map $j : S^1 \to \mathbb{R}^2 - \{0\}$ then we'll have a contradiction.

Consider $F(x,t) = tx + (1-t)w(x)$

$x \in S^1$. $F(x,0) = w(x)$ and $F(x,1) = x = j(x)$

$\Rightarrow$ if $F \neq 0$ then F is the required homotopy.

$F(x,t) \neq 0$ for $t=0$ and $t=1$.

If $F(x,t) = 0$ for some t w/ $0<t<1$

$$\Rightarrow \quad tx + (1-t)w(x) = 0 \quad \Rightarrow \quad w(x) = \underbrace{\left(\frac{t}{t-1}\right)}_{<0} x$$

$\Rightarrow$ $w(x)$ points directly inward at x.

$\Rightarrow$ $F(x,t) \neq 0$ $\Rightarrow$ we have a contradiction.

$\therefore$ $\exists$ a point $x \in S^1$ where $v(x)$ point directly inward. $\blacksquare$

<u>Thm</u> (Brouwer fixed point theorem for $n=2$/ for the disc).

If $f : B^2 \to B^2$ is continuous, then $\exists$ a fixed point of f i.e., $\exists\ x \in B^2$ s.t. $f(x) = x$.

<u>Proof</u>:- We prove by contradiction, i.e, suppose $\nexists$ any $x \in B^2$ s.t $f(x) = x$.

define $v(x) = f(x) - x$ is a nonvanishing v.f. $(x, v(x))$ on B^2. From the previous thm $\exists\, x \in S^1$ s.t. $f(x) - x = ax$ for some $a > 0$.

$\Rightarrow$ $f(x) = ax + x = (1+a)x$

is a contradiction b/c $f(x) = (1+a)x$ would lie outside the unit ball.

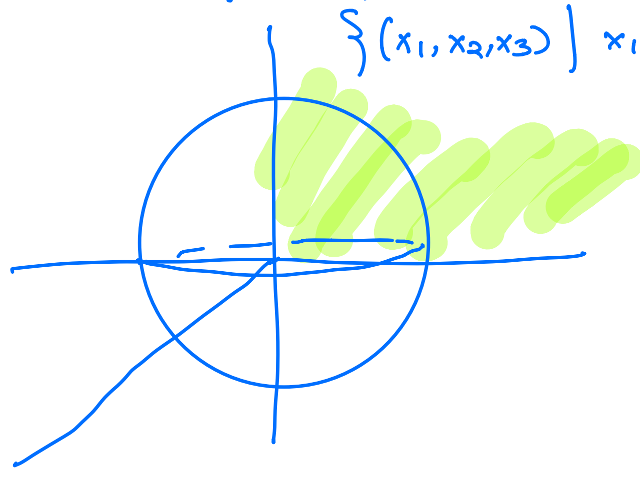
$\Rightarrow$ $v(x) = 0$ for some $x \in B^2$

$\Rightarrow$ $f(x) = x$. $\boxtimes$

Cor: Let A be a 3×3 matrix of positive real numbers. Then A has a positive real eigenvalue.

Proof: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map whose matrix is A.

B = intersection of S^2 w/ the 1st octant of $\mathbb{R}^3$
$\{(x_1, x_2, x_3) \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$



B is homeomorphic to B^2. $\Rightarrow$ the fixed point thm holds for cont. maps of $B \to B$.

If $\overset{(x_1,x_2,x_3)}{\overset{\shortparallel}{x}} \in B$ then all components are non-negative and atleast one is positive.

$\Rightarrow$ $T(x) \in \mathbb{R}^3$ all of whose components are positive

$$\Rightarrow \quad x \longmapsto \frac{T(x)}{\|T(x)\|} : B \longrightarrow B$$

is a cont. map $\Rightarrow$ $\exists$ x_0 s.t. $\dfrac{T(x_0)}{\|T(x_0)\|} = x_0$

$$\Rightarrow \quad T(x_0) = \underbrace{\|T(x_0)\|}_{} x_0 \qquad \left(Tx = \lambda x, \; \begin{matrix} x \text{ eigenvector} \\ \lambda \text{ eigenvalue} \end{matrix} \right)$$

real positive eigenvalue of T or A.

$\blacksquare$

Borsuk-Ulam Theorem (problem set 5)

Given a continuous map $f: S^2 \to \mathbb{R}^2$ $\exists$ $x \in S^2$ s.t.
$f(x) = f(-x)$.

The fundamental theorem of Algebra

A polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

of degree n w/ real or complex coefficients has at least one root.

Proof: $f: S^1 \to S^1$, $f(z) = z^n$, $n \in \mathbb{Z}_+$

$f_*: \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ of fundamental groups is injective.

Let $p_0: I \to S^1$ standard loop of S^1

$$p_0(s) = e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s)$$

image of $p_0(s)$ under f_*

$$f(p_0(s)) = (e^{2\pi i s})^n = (\cos 2\pi n s, \sin 2\pi n s)$$

This loop lifts to the path $s \mapsto ns$ in $\mathbb{R}$.

$\therefore$ viewing this map in $\mathbb{R}$,

f_* is just multiplication by n

$\Rightarrow$ f_* is injective.

if $g: S^1 \to \mathbb{R}^2 - \{0\}$

$g(z) = z^n$ then g is not nullhomotopic.

$$g = j \circ f$$

$j: S^1 \to \mathbb{R}^2 - \{0\}$.

$\because$ f_* is inj. and j_* is injective. $\Rightarrow$ $g_* = j_* \circ f_*$ is injective $\Rightarrow$ g is not nullhomotopic.

Given $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$

assume $|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$.

and show that the eqn has a root lying in B^2.

Suppose not.

$$k : B^2 \longrightarrow \mathbb{R}^2 - \{0\} \text{ by}$$

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let $h = k|_{S^1} : S^1 \to \mathbb{R}^2 - \{0\}$

$\Rightarrow$ h is nullhomotopic (from the previous thm.)

We'll get a contradiction by producing a homotopy b/w h and g.

define $F : S^1 \times I \to \mathbb{R}^2 - \{0\}$

$$F(z,t) = z^n + t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)$$

$$F(z,0) = z^n = g$$

$$F(z,1) = h$$

Claim :- $F(z,t) \neq 0$.

$$|F(z,t)| \geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)|$$

$$\geq 1 - t\left(|a_{n-1} + z^{n-1} + \cdots + a_1 z + a_0|\right)$$

$$\geq 1 - t\left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|\right)$$

$$> 0$$

$\therefore$ $F \neq 0$. $\Rightarrow$ F is the required homotopy

$\Rightarrow$ we get a contradiction.

$\therefore$ $\exists$ a root to $z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

for the general case,

Choose $c \in \mathbb{R}$, $c > 0$ large enough w/ $z = cy$

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0$$

$$\Downarrow$$

$$y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_1}{c^{n-1}} y + \frac{a_0}{c^n} = 0$$

$\blacksquare$