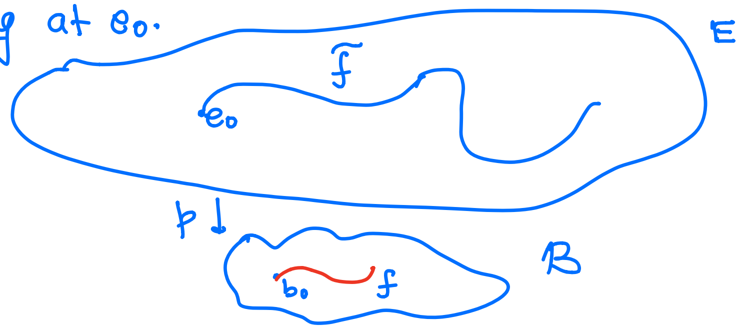
Lecture 13

Recall :-

$$\begin{tikzcd} & E \arrow[d, "p"'] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

$\tilde{f} : X \to E$ is a lift of f if $p \circ \tilde{f} = f$.

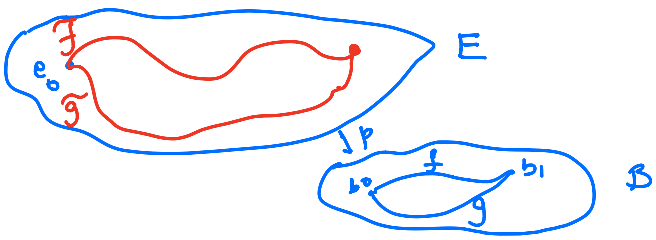
Path-lifting lemma $p : E \to B$ covering map w/ $p(e_0) = b_0$. Any path $f : [0,1] \to B$ at b_0 has a unique lift to a path $\tilde{f}$ in E starting at e_0.



Homotopy-Lifting lemma

$p : E \to B$ covering map w/ $p(e_0) = b_0$. Let $F : I \times I \to B$ be continuous w/ $F(0,0) = b_0$. $\exists!$ lift of F to a continuous map $\tilde{F} : I \times I \to E$ w/ $\tilde{F}(0,0) = e_0$. If F is a path homotopy then so is $\tilde{F}$.

Thm $p : E \to B$ covering map w/ $p(e_0) = b_0$. Let f, g be two path in B from b_0 to b_1 and $\tilde{f}, \tilde{g}$ be lifts in E, starting at e_0. If $f \simeq_p g$ then $\tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f} \simeq_p \tilde{g}$.



<u>Def^n</u> Let $p: E \to B$ covering map, $b_0 \in B$, $p(e_0) = b_0$. $[f] \in \pi_1(B, b_0)$, $\tilde{f}$ be the lift of f to a path in E, beginning in e_0. Let $\phi([f])$ be the end point $\tilde{f}(1)$ of $\tilde{f}$.

$$\phi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

ϕ - lifting correspondence.

<u>Theorem</u> Let $p: E \to B$ covering map, $p(e_0) = b_0$. If E is path connected, then $\phi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$ is surjective. If E is simply connected, it is bijective.

<u>Proof</u>:- If E is path connected and $e_1 \in p^{-1}(b_0)$ then $\exists$ a path $\tilde{f}$ in E from e_0 to e_1. $f = p \circ \tilde{f}$ is a loop in B at b_0. $\Rightarrow$ $[f] \in \pi_1(B, b_0)$ and

$$\phi([f]) = e_1 \Rightarrow \phi \text{ is surjective.}$$

Suppose E is simply connected. Let $[f]$ and $[g] \in \pi_1(B, b_0)$ and let $\phi([f]) = \phi([g]) \Rightarrow \tilde{f}(1) = \tilde{g}(1)$, $\tilde{f}$ and $\tilde{g}$ begin at e_0.

$\exists$ a path homotopy $\tilde{F}$ b/w $\tilde{f}$ and $\tilde{g}$ $\Rightarrow$

$F = p \circ \tilde{F}$ is a path hom. b/w f and g in B

$\Rightarrow [f] = [g] \Rightarrow \phi$ is a bijection. $\blacksquare$

Thm $\pi_1(S^1) \cong (\mathbb{Z}, +)$.

Proof: $p: \mathbb{R} \to S^1$ be the covering map

$p(x) = (\cos 2\pi x, \sin 2\pi x)$

let $e_0 = 0$ in $\mathbb{R}$ w/ $b_0 = (1,0)$ in S^1.

$p^{-1}(b_0) = \mathbb{Z} \Rightarrow$ by previous thm

$\phi : \pi_1(S^1, b_0) \longrightarrow \mathbb{Z}$ is a bijection.

$$\phi\left([f] * [g]\right) = \underset{\underset{n}{\parallel}}{\phi([f])} + \underset{\underset{m}{\parallel}}{\phi([g])} \quad \text{(Want)} \longrightarrow 1$$

$\tilde{f}$ lift of f, $\tilde{g}$ lift of g. and $\tilde{f}(1) = n$
$\tilde{g}(1) = m$

$\tilde{\tilde{g}}(x) = n + \tilde{g}(x)$ path in $\mathbb{R}$.

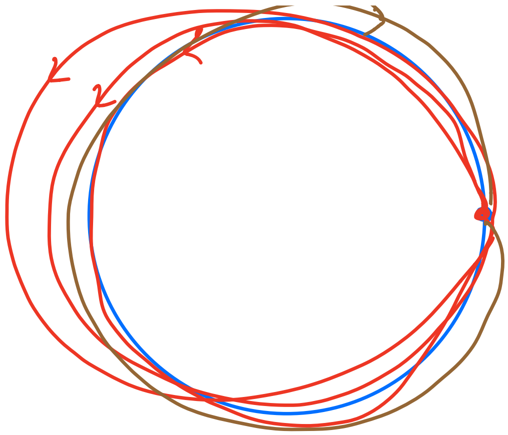
$\because p(n+x) = p(x) \ \forall x \in \mathbb{R} \Rightarrow \tilde{\tilde{g}}$ is a lift of g

$\tilde{\tilde{g}}(0) = n \Rightarrow \tilde{f} * \tilde{\tilde{g}}$ is defined and it is the lift of $f * g$ which begins at 0.

$\tilde{\tilde{g}}(1) = n + \tilde{g}(1) = n+m$

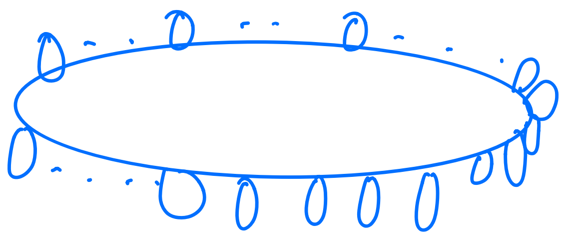
$\phi\left([f] * [g]\right) = n+m = \phi([f]) + \phi([g])$, proves 1

$\square$



$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

$\pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

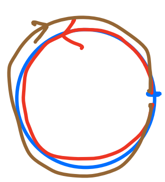


$\langle x \rangle = \{ x^n \mid n \in \mathbb{Z} \}$, x generator of $\langle x \rangle$

$\downarrow$ cylic group. $|\langle x \rangle|$ = order of the group.

$y \cdot z = x^n \cdot x^m = x^{n+m}$

$y^{-1} = x^{-n}$ $\qquad e = x^0$.



A cylic group of order $k \cong \mathbb{Z}_k$ or $\mathbb{Z}/k$

of group of integers modulo k.

Theorem:- Let $p: E \to B$ is a covering map w/ $p(e_0) = b_0$.

a) The hom. $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

b) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map.

$$\Phi : \pi_1(B, b_0)/H \longrightarrow p^{-1}(b_0).$$

Φ is bijective if E is path connected.

c) If f is a loop based at b_0 then $[f] \in H$

$\iff$ f lifts to a loop in E based at e_0.

Digression:- G, $H \leq G$

set of left cosets of H in G

$\{H, g_1H, g_2H, \ldots, g_nH, \ldots\}$

$gH = \{g \cdot h \mid h \in H\}$. not always a subgroup of G.

Proof:

a) Let $\tilde{h}$ be a loop based at e_0 s.t $p_*([\tilde{h}]) = [e_{b_0}] \in \pi_1(B, b_0)$.

F is a path homotopy b/w $p \circ \tilde{h}$ and e_{b_0}.

If $\tilde{F}$ is the lift of F in E w/ $\tilde{F}(0,0) = e_0$

$\Rightarrow$ $\tilde{F}$ is a path hom. b/w $\tilde{h}$ and e_{e_0}

$\Rightarrow$ $[\tilde{h}] = [e_{e_0}]$ is the identity element $\pi_1(E, e_0)$

$\Rightarrow$ p_* is injective.

(b). f, g loops in B at b_0

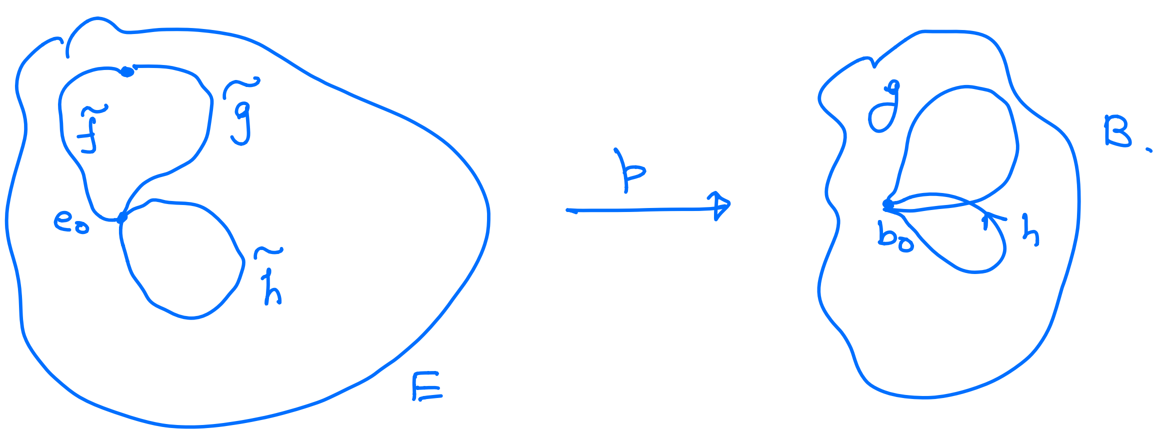
$\tilde{f}$ $\tilde{g}$ begin at e_0.

$\phi([f]) = \tilde{f}(1)$, $\phi([g]) = \tilde{g}(1)$.

We'll prove that $\phi([f]) = \phi([g]) \Leftrightarrow [f] \in H * [g]$.

$\Leftarrow$ Let $[f] \in H * [g] \Rightarrow [f] = [h * g]$

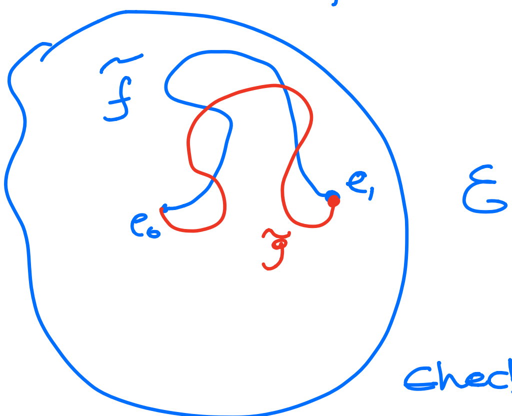
where $h = p \circ \tilde{h}$, $\tilde{h}$ some loop in E based at e_0.



$\tilde{h} * \tilde{g}$ is defined, it is a lift of $h * g$

$\because$ $[f] = [h * g]$ $\Rightarrow$ $\tilde{f}$ and $\tilde{h} * \tilde{g}$ must end at the same point. $\Rightarrow$ $\phi([f]) = \phi([g])$.

$\Longrightarrow$. Let $\phi([f]) = \phi([g]) \Rightarrow \tilde{f}$ and $\tilde{g}$ must have the same end point in $\mathcal{E}$.



$\tilde{f} * \tilde{g}^{-1} = \tilde{h}$ which is a loop based at e_0.

Check $[\tilde{h} * \tilde{g}] = [\tilde{f}]$

if $\tilde{F}$ is a path homotopy in $\mathcal{E}$ b/w $\tilde{h} * \tilde{g}$ and $\tilde{f} \Longrightarrow p \circ \tilde{F}$ is a path hom. in B b/w $h * g$ and f where $h = p \circ \tilde{h}$

$\Rightarrow [f] = [h * g]$ where $h \in p_*(\pi_1(E, e_0))$.

$\Rightarrow [f] \in H * [g]$ $\boxed{A}$

If E is path-connected $\Rightarrow \Phi$ is surjective. as well

$\Rightarrow \Phi$ is a bijection.

c) Exercise.

$\boxed{\text{VI}}$

Defn If $A \subset X$, a retraction of X onto A is a continuous map $r : X \longrightarrow A$ s.t. $r|_A = id_A$.

If such a r exists then we say A is a retract of X.

Thm:- (PSET 4)

$j: A \to X$ inclusion, A is a retract of X.

then the homomorphism induced by j is injective

i.e., $j_*: \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an injection.

Thm. (PSET4)

There is no retraction of B^2 onto S^1.

Theorem:- Let $h: S^1 \to X$ be a continuous map. Then TFAE:-

1) h is nullhomotopic.
2) h extends to a continuous map $k: B^2 \to X$.
3) h_* is the trivial hom. of fundamental groups.

i.e., $h_*([f]) = [e]$ in $\pi_1(X, x_0)$.

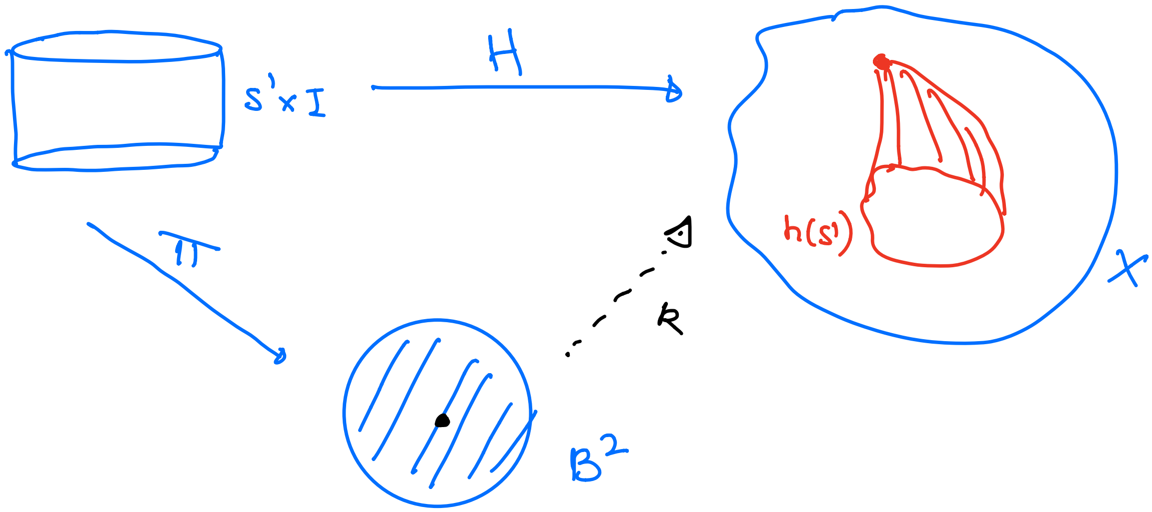
Proof

1) $\Rightarrow$ 2)

Let $H: S^1 \times I \to X$ be a homotopy b/w h and a constant.

$\pi: S^1 \times I \to B^2$ given by

$\pi(x, t) = (1-t)x$



Check :- π is a quotient map. It is injective apart from $S^1 \times I$ $\Rightarrow$ from the thm on continuous functions on quotient spaces $\exists$ a map $k : B^2 \to X$ continuous and $k|_{S^1} = h$.

2) $\Rightarrow$ 3).

$j : S^1 \to B^2$ inclusion $j(x) = x \in B^2$

$k : B^2 \to X$ $\quad h = k|_{S^1}$

$\Rightarrow$ $h : S^1 \to X$, $h = k \circ j$. By functorial prop. of the fundament group,

$$h_* = k_* \circ j_* : \pi_1(S^1, b_0) \to \pi_1(X, x_0)$$

$j_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0)$. is trivial as $\pi_1(B^2)$ is $\Rightarrow$ h_* is trivial.

(3) $\Rightarrow$ (1) Let $p : \mathbb{R} \to S^1$ be the usual covering map and let $p_0 : I \to S^1$ be $p|_I$.

Then as we discussed, $[p_0]$ generates the cyclic group $\pi_1(S^1, b_0)$ as $\tilde{p}_0$ starts at 0 and ends at 1.

Let $x_0 = h(b_0)$.

$\because$ $h_* : \pi_1(S^1, b_0) \longrightarrow \pi_1(X, x_0)$ is trivial

$\Rightarrow$ the loop $[f] = [h \circ p_0]$ is the identity element of $\pi_1(X, x_0)$.

$\therefore$ $\exists$ a path hom. in X, F b/w f and e_{x_0}.

We note that $p_0 \times id : I \times I \longrightarrow S^1 \times I$ is a quotient map which is injective apart from

$$\left.\begin{matrix} 0 \times t \\ 1 \times t \end{matrix}\right\} \longrightarrow b_0 \times t \qquad \forall\ t \in I.$$

moreover $F(0 \times I) = F(1 \times I) = F(I \times 1) = x_0 \in X$

$\therefore$ from the theorem on continuous maps of quotient spaces $\exists$ a continuous map $H : S^1 \times I \longrightarrow X$ which is a homotopy b/w h and a constant map.

$\Rightarrow$ h is nullhomotopic. ∎