Lecture 12

* Prob. Set 4 due on 01/06/21.

Recall:– $p : E \to B$ continuous surjective map.

$U \underset{\text{open}}{\subset} B$ is evenly covered by p if

$$p^{-1}(U) = \bigsqcup V_\alpha \quad , \quad V_\alpha \underset{\text{open}}{\subset} E$$

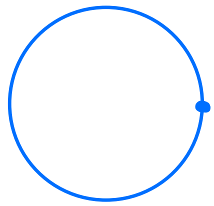
$p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

$\{V_\alpha\}$ partition of $p^{-1}(U)$ into slices.

$p : E \to B$ cont., surjective. If $\forall\, b \in B\ \exists\, U \ni b$, $U \underset{\text{open}}{\subset} B$ which is evenly covered by p then p is a covering map, E cover or a covering space of B.

$$p : \mathbb{R} \to S^1$$
$$p(x) = (\cos 2\pi x, \sin 2\pi x) \quad \text{covering map.}$$

$p \times p' : E \times E' \longrightarrow B \times B'$ is also a covering map if $p : E \to B$ and $p' : E' \to B'$ is a covering map.



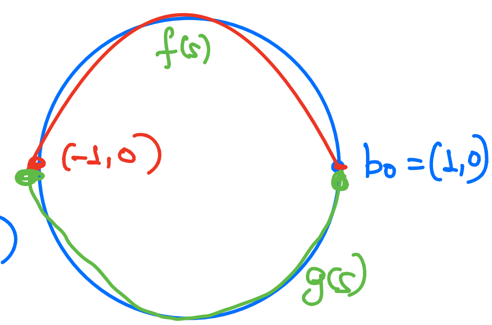
Defn $p: E \to B$. a lifting of f is a map $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

$p: \mathbb{R} \to S^1$ as above.

path $f: [0,1] \to S^1$

$f(0) = b_0$ s.t. $f(s) = (\cos \pi s, \sin \pi s)$



$\exists$ lift of f, $\tilde{f}: [0,1] \to \mathbb{R}$

$\tilde{f}(s) = s/2$ begins at 0 and ends at $1/2$.

$g(s): [0,1] \to S^1$, $g(s) = (\cos \pi s, -\sin \pi s)$

$\tilde{g}(s) = -s/2$ begins at 0 and ends at $-1/2$.

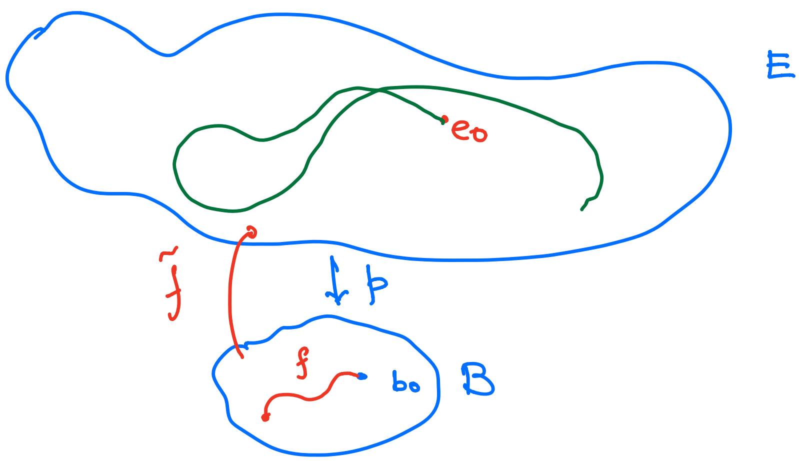
$h(s) = (\cos 4\pi s, \sin 4\pi s)$

$\tilde{h}(s) = 2s$ begins at 0 and ends at 2.

Lemma (path-lifting Lemma)

Let $p: E \to B$ be a covering map, $p(e_0) = b_0$.

Any path $f: [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.



Existence

Proof: Cover B by open sets U each of which are evenly covered by p.

$[0,1]$ is compact $\leadsto$ look at a subdivision

$s_0, s_1, s_2, \dots, s_n$ $\quad [0, s_0] \cup [s_0, s_1] \cup \dots \cup [s_{n-1}, s_n]$

$\Rightarrow$ $f([s_i, s_{i+1}])$ lie in an open set U as above.

We define $\tilde{f}$ step by step.

define, $\tilde{f}(0) = e_0$. Suppose $\tilde{f}(s)$ is defined for $0 \leq s \leq s_i$.

define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows.

$f([s_i, s_{i+1}]) \subset U \underset{\text{open}}{\subset} B$ evenly covered by p.

$\{V_\alpha\}$ be a partition of $p^{-1}(U)$.

$$p|_{V_\alpha} : V_\alpha \to U \quad \text{is a homeomorphism } \forall \alpha.$$

$\tilde{f}(s_i)$ lies in one of the V_α's, say V_0.

Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$

$$\tilde{f}(s) = \left(p|_{V_0}\right)^{-1}\left(f(s)\right).$$

$\because$ $p|_{V_0}$ is a homeomorphism $\Rightarrow$ $\tilde{f}$ is continuous on $[s_i, s_{i+1}]$.

Continue in this way and define $\tilde{f}$ on all of $[0,1]$.

$\tilde{f}$ is continuous. and the way we have defined $\tilde{f}$, we get $\quad p \circ \tilde{f} = f$.

$\Rightarrow$ $\tilde{f}$ is a lift of f.

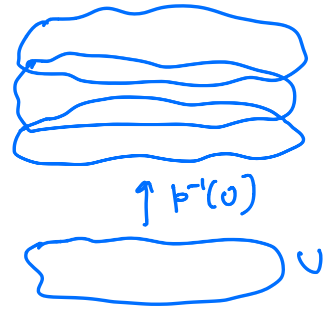
<u>Uniqueness</u>

Suppose $\tilde{\tilde{f}}$ is another lifting of f begining at e_0. $\tilde{\tilde{f}}(0) = \tilde{f}(0) = e_0$.

Suppose $\tilde{\tilde{f}}(s) = \tilde{f}(s)$, $\quad 0 \le s \le s_i$

$\because$ $f([s_i, s_{i+1}]) \subset U \underset{\text{open}}{\subset} B$, $\tilde{\tilde{f}}(s)$ is a lift of f , i.e, $\quad p \circ \tilde{\tilde{f}} = f$

$$\Rightarrow \quad \underbrace{\tilde{\tilde{f}}([s_i, s_{i+1}])}_{\text{connected}} \text{ must lie in } p^{-1}(U) = \bigsqcup V_\alpha .$$



$$\Downarrow$$

it must lie entirely in one of V_α.

But, $\tilde{\tilde{f}}(s_i) = \tilde{f}(s_i) \in V_0$

$$\Rightarrow \quad \tilde{\tilde{f}}([s_i, s_{i+1}]) \subset V_0 .$$

$\therefore$ If $s \in [s_i, s_{i+1}]$, $\tilde{\tilde{f}}(s) = y \in V_0$ lying in $p^{-1}(f(s))$. But y is unique and is given by

$$(p|_{V_0})^{-1}(f(s)) = \tilde{f}(s) = \tilde{\tilde{f}}(s), \quad \forall\, s \in [s_i, s_{i+1}]$$

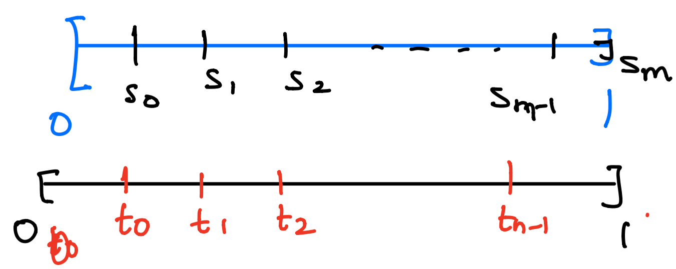
$$\Rightarrow \quad \tilde{\tilde{f}} = \tilde{f}$$

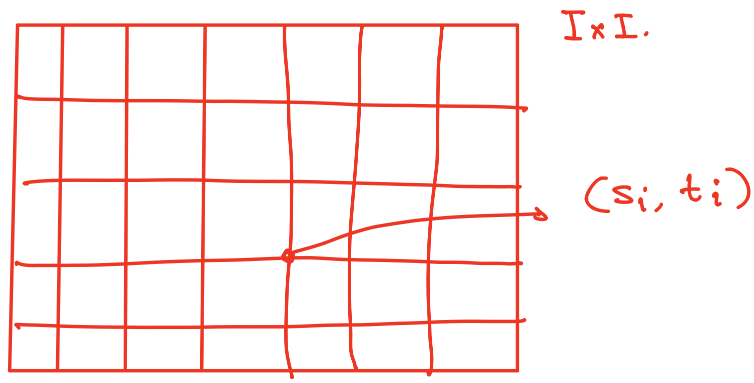
Lemma (Homotopy-lifting lemma)

Let $p: E \to B$ be a covering map, $p(e_0) = b_0$.
Let $F: I \times I \to B$ be continuous w/ $F(0,0) = b_0$.
There is a unique lifting of F to a continuous map $\tilde{F}: I \times I \longrightarrow E$ s.t. $\tilde{F}(0,0) = e_0$.

If F is a path homotopy, then so is $\tilde{F}$.

Proof:





$\tilde{F}(0,0) = e_0$. By the previous lemma, extend $\tilde{F}$ to the left edge $0 \times I$ and the bottom edge $I \times 0$.

$$I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1} \times t_j]$$

$\hookrightarrow$ is mapped by F into an open set U of B which is evenly covered by p.

Suppose $\tilde{F}$ is alredy defined on $A \subset I \times I$.

$A = 0 \times I \cup I \times 0 \cup$ all previous $I_{i_0} \times J_{j_0}$ $\quad (i_0, j_0) \in I \times I$

$\downarrow$

those rectangles $I_i \times J_j$ s.t. $j < j_0$ or if $j = j_0$ then $i < i_0$.

Assume $\tilde{F}$ is a continuous lift of $F|_A$.

Define $\tilde{F}$ on $I_{i_0} \times J_{j_0}$ as follows.

Suppose $F(I_{i_0} \times J_{j_0}) \subset \underset{\text{open}}{U} \subset B$

Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$.

$\tilde{F}$ is already defined on $A \cap (I_{i_0} \times J_{j_0})$

$\Rightarrow$ $\tilde{F}(A \cap (I_{i_0} \times J_{j_0}))$ must be connected inside

$\bigsqcup V_\alpha$, but $\tilde{F}(i_0, j_0) \in V_0$ (say)

$\Rightarrow$ $\tilde{F}(A \cap (I_{i_0} \times J_{j_0})) \subset V_0$

$$\tilde{F}(x) = \left(p|_{V_0}\right)^{-1}(F(x))$$

Continuity of $\tilde{F}$ and uniqueness follows via the same reasoning as before.

$p \circ \tilde{F} = F$ by def^n $\Rightarrow$ $\tilde{F}$ is a lift.

Suppose F is a path homotopy in B.

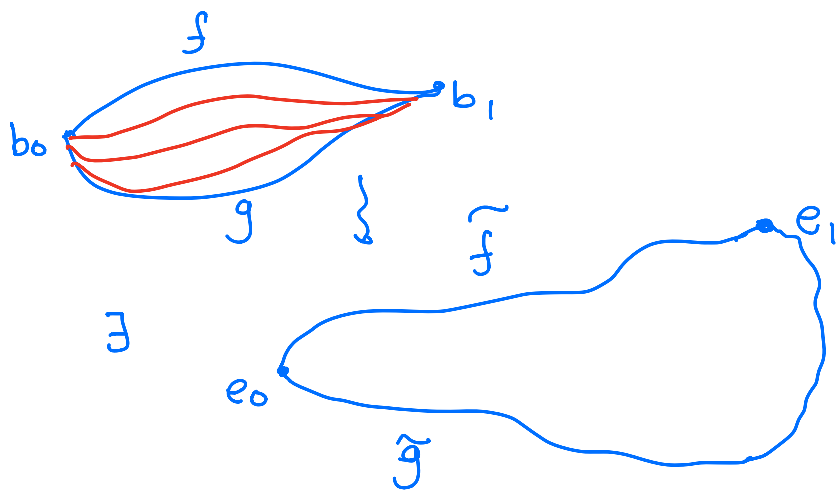
$F(0 \times I) = b_0$ and $\because$ $\tilde{F}$ is a lift of F $\Rightarrow$

$\tilde{F}(0 \times I) = p^{-1}(b_0) = \{e_0\}$ as $p^{-1}(b_0)$ has discrete topology in E as $\tilde{F}(0\times I)$ is connected.

Similarly $\tilde{F}(1\times I) = \{e_1\}$

$\Rightarrow$ $\tilde{F}$ is a path-homotopy. $\square$

<u>Theorem</u>: Let $p: E \to B$ be a covering map, $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1.



If f and g are path-homotopic then $\tilde{f}$ and $\tilde{g}$ end at the same point and are path homotopic.

<u>Proof</u> $\tilde{F}(s,0) = \tilde{f}(s)$ $\forall$ $s \in I$

$\tilde{F}|_{I\times 1}$ path in E which is a lift of $F|_{I\times 1}$

$\Rightarrow$ $\tilde{F}|_{I\times 1}$ starts at e_0, by uniqueness of path lift $\tilde{F}(s,1) = \tilde{g}(s)$

$\Rightarrow$ both $\tilde{f}$ and $\tilde{g}$ end at e_1 and $\tilde{F}$ is the required path homotopy. $\boxtimes$

Def^n $p: E \to B$ is a covering map w/ $b_0 \in B$. Let $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lift of f to a path in E starting at e_0.

Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$.

We get a map

$$\phi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

↳ lifting correspondence from the covering map p.

↳ ϕ depends on the choice of e_0.

Theorem Let $p: E \to B$ cov. map, $p(e_0) = b_0$.
If E is path connected then

$$\phi: \pi_1(B, b_0) \to p^{-1}(b_0) \text{ is surjective.}$$

If E is simply connected then ϕ is bijective.

Theorem (Fundamental group of S^1).
The fundamental group of $S^1 \cong (\mathbb{Z}, +)$.

Proof. Consider $p: \mathbb{R} \to S^1$ w/ $p(x) = (\cos 2\pi x, \sin 2\pi x)$

covering map. Let $e_0 = 0$. $b_0 = (1,0)$.

$$p^{-1}(b_0) = p^{-1}((1,0)) = \mathbb{Z}$$

$\because$ $\mathbb{R}$ is simply connected $\Rightarrow$

$$\phi : \pi_1(S^1, b_0) \longrightarrow \mathbb{Z} \text{ is bijective.}$$

To show that ϕ is a group homomorphism.

Let $[f]$, $[g] \in \pi_1(S^1, b_0)$.

$\tilde{f}$ $\tilde{g}$ be lifts in $\mathbb{R}$ beginning at 0.

Let $n = \tilde{f}(1)$, $m = \tilde{g}(1)$.

$\Rightarrow$ $\phi([f]) = n$, $\phi([g]) = m$ (by def^n).

Suppose $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = n + \tilde{g}(s) \text{ in } \mathbb{R}.$$

$\because$ $p(n+x) = p(x)$ $\forall$ $x \in \mathbb{R}$

$\Rightarrow$ $\tilde{\tilde{g}}$ is a lift of g under p

and $\tilde{\tilde{g}}$ begins at n.

$\Rightarrow$ $\tilde{f} * \tilde{\tilde{g}}$ is defined. and it is the lift

of $f * g$ which begins at 0.

The end point of the path $\widetilde{\widetilde{g}}$, is $\widetilde{g}(1) = n+m$

$\therefore$ by def^n of ϕ

$$\phi\left([f] * [g]\right) = n+m = \phi([f]) + \phi([g])$$

$\Rightarrow$ ϕ is a homomorphism

$$\Rightarrow \quad \pi_1(S^1) \cong (\mathbb{Z}, +)$$