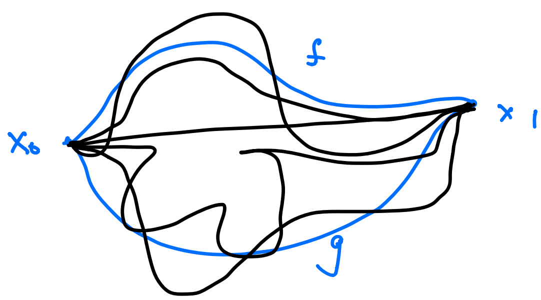
Lecture 11

Recall:- $f, g : X \to Y$ are homotopic ($f \simeq g$) if $\exists\ F : X \times I \to Y$ ($I = [0,1]$) continuous s.t.

$$F(x,0) = f(x), \quad F(x,1) = g(x) \qquad \forall\, x \in X.$$

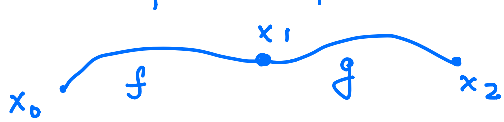
$f, g : I \to X$ are paths in X, joining x_0 and x_1. They are path-homotopic ($f \simeq_p g$) if $\exists\ F : I \times I \to X$ continuous s.t.

$$F(s,0) = f(s), \quad F(s,1) = g(s)$$
$$F(0,t) = x_0, \quad F(1,t) = x_1 \qquad \forall\, s,t \in I.$$



• $\simeq$ and $\simeq_p$ are equivalence relations.

For



The concatenation of f and g is a path h from x_0 to x_2

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, 1/2] \\ g(2s-1) & \text{for } s \in [1/2, 1] \end{cases} = f * g$$

$$[f]*[g] = [f*g]$$

$\pi_1(X, x_0)$ = fundamental group of X relative to the base point x_0

$= \{ [\gamma] \mid \gamma \text{ is a loop at } x_0 \}$

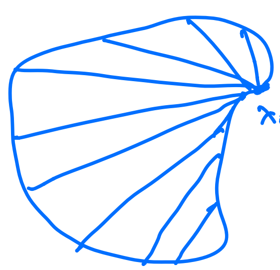
e_{x_0} = constant loop at x_0

*

$$[f]^{-1} = [f^{-1}]$$

<u>Remark</u> :- $\pi_1(X, x_0)$ is a group. $[e_{x_0}]$

$$\pi_1(\mathbb{R}^n, x_0) = \{[e_{x_0}]\} = 0.$$



γ in $\mathbb{R}^n$

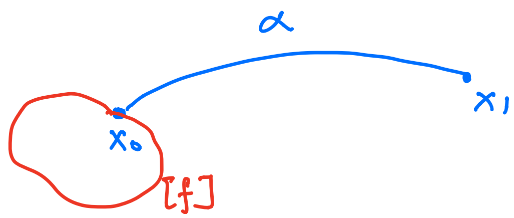
$\because$ straight-line homotopy proves that any two paths in $\mathbb{R}^n$ are path-homotopic $\Rightarrow$ there is only one equivalence class $\Rightarrow \pi_1(\mathbb{R}^n, x_0) = 0$.

Any convex subset of $\mathbb{R}^n$, then its fundamental group is again trivial.

The unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$

$\quad = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ has trivial fundamental group.

Suppose $x_1 \in X$. Let α be a path in X joining x_0 and x_1.



Define a map

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

$$\hat{\alpha}([f]) = [\alpha]^{-1} * [f] * [\alpha]$$

is a well-defined map as $*$ is well-defined.

Theorem:- $\hat{\alpha}$ is an isomorphism of groups.

$\varphi : G \longrightarrow G'$ is a homomorphism if $\varphi(g \cdot h) = \varphi(g) \cdot \varphi(h)$.

φ is an isomorphism if it is a bijective homomorphism.

Proof. $\hat{\alpha}$ is a homomorphism as,

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = \left([\alpha]^{-1} * [f] * [\alpha]\right) * \left([\alpha]^{-1} * [g] * [\alpha]\right)$$

$$= [\alpha]^{-1} * [f] * [g] * [\alpha]$$
$$= \hat{\alpha}\left([f] * [g]\right)$$

$$\hat{\beta} : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$
$$\hat{\beta}([h]) = [\alpha] * [h] * [\alpha]^{-1}$$

$\hat{\beta}$ is an inverse to $\hat{\alpha}$. $\Rightarrow$ $\hat{\alpha}$ is an isomorphism. $\boxtimes$

Corr :- If X is path connected then the fundamental groups relative to any point in X are all isomorphic.

Exer. The iso. b/w $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is independent of the path $\Leftrightarrow$ the fundamental group is abelian.

Defn X is simply-connected if it is path-connected and if $\pi_1(X, x_0) = 0$.

Suppose $h: X \to Y$ is a continuous map s.t $h(x_0) = y_0$. $h: (X, x_0) \longrightarrow (Y, y_0)$

If f is a loop based at x_0 then $h \circ f : I \to Y$ is a loop in Y based at y_0.

Def. $h : (X, x_0) \to (Y, y_0)$ is a continuous map. Define

$$(h_{x_0})_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

$$(h_{x_0})_*([f]) = [h \circ f]$$

$(h_{x_0})_*$ is called the homomorphism induced by h.

$(h_{x_0})_*$ is well-defined. Let f and $f' \in [f]$. $\Rightarrow$

F is a path homotopy b/w f and f'.

$\Rightarrow$ $h \circ F$ is a path-homotopy b/w $h \circ f$ and $h \circ f'$

and $(h \circ f) * (h \circ g) = h \circ (f * g)$

$\therefore$ the homomorphism is well-defined.

Theorem (Induced homomorphism satisfy "functorial properties").

1) If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous. Then

$$(k \circ h)_* = k_* \circ h_*$$

2) If $i : (X, x_0) \to (X, x_0)$ is ~~the~~ identity map then i_* is the identity hom.

Proof:- $(k \circ h)_*([f]) = [(k \circ h) \circ f]$

$\parallel$

$$(k_* \circ h_*)([f]) = k_*(h_*[f])$$
$$= k_*([h \circ f]) = [k \circ (h \circ f)]$$

$$i_*([f]) = [i \circ f] = [f] \Rightarrow i_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$
is the identity hom. $\boxtimes$

Corr. (fundamental group is a top. invariant)

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

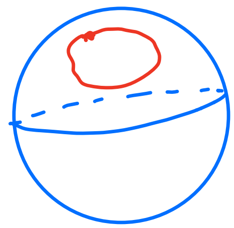
Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h.

Then
$$k_* \circ h_* = (k \circ h)_* = (i)_* = i_* \quad \text{and}$$
$$h_* \circ k_* = (h \circ k)_* = (j)_* = j_*$$
identity map of (Y, y_0).

$\Rightarrow h_*$ is an isomorphism w/ inverse k_* $\boxtimes$

Remarks :-

1) Converse is NOT true :- isomorphic fundamental groups $\not\Rightarrow$ homeomorphic spaces. e.g. $\mathbb{R}$, $\mathbb{R}^2$
S^n, S^m, $n \neq m$.

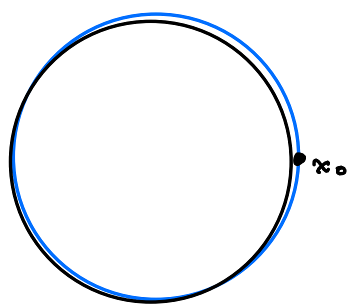


$\pi_1(S^n, x_0) = 0$, $n \geq 2$.

$\pi_1(S^1, x_0) = \mathbb{Z}$

2) Contrapositive is very useful.

non-isomorphic fundamental groups $\Rightarrow$ non-hom. spaces.



Covering Spaces (cf. Riemann surfaces, complex geom.)

Def. Let $p : E \to B$ is a continuous surjective map.

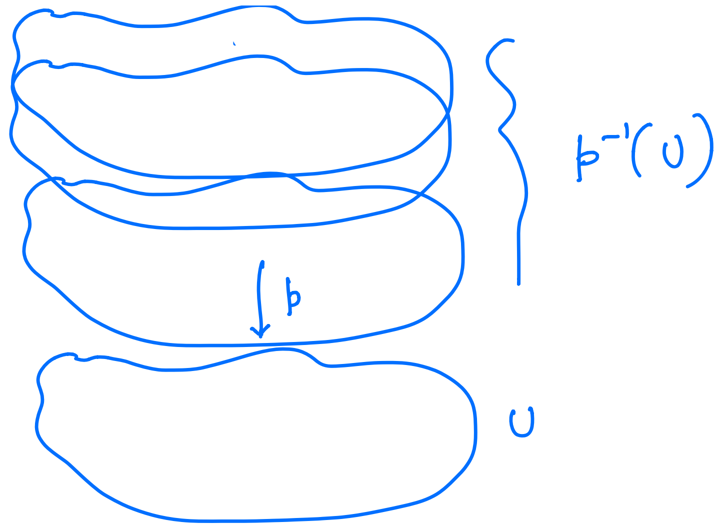
$U \underset{\text{open}}{\subset} B$ is evenly covered by p if

$$p^{-1}(U) = \bigsqcup V_\alpha$$

$$V_\alpha \underset{\text{open}}{\subset} E$$

and $\forall \alpha$ $p|_{V_\alpha} \to U$ is a homeomorphism.

$\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into slices.



Def^n Let $p: E \to B$ is continuous & surjective.
If every $b \in B$ has a nbd U that is evenly covered by p, then p is a covering map and E is a covering space of B.

If $p: E \to B$ is a covering map then $\forall\, b \in B$, $p^{-1}(b)$ in E has discrete topology.

Ex. 1) $i: X \to X$ identity map is always a covering map.

2) $E = X \times \{1, 2, \ldots, n\}$
$p: E \to X$ w/ $p(x, i) = x$ $\forall\, i$ is a covering map.

3) $p: \mathbb{R} \longrightarrow S^1$ given by

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map. $\mathbb{R}$ is a covering space of S^1.

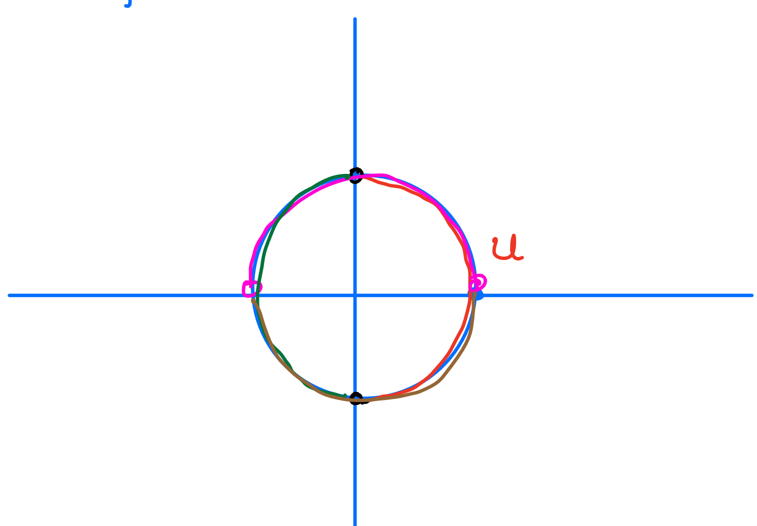
Remarks:- 1) A space can have multiple covering space and any covering space which is simply-connected is known as a universal cover.

$\mathbb{R}$ is a universal cover of S^1.

$[n, n+1]$

$$p^{-1}(U) = \bigsqcup V_n \quad , \quad V_n = \left(n - \tfrac{1}{4}, n + \tfrac{1}{4}\right)$$

each $V_n \underset{\text{open}}{\subset} \mathbb{R}$ is mapped homeomorphically via p to U.



We can do the same thing for all the four nbds described $\Rightarrow$ $\mathbb{R}$ is a covering space of S^1.

4) $p : S^1 \longrightarrow S^1$ (e.g. Riemann surfaces).

$p(z) = z^n$, $n \in \mathbb{Z}_+$ is a covering map.

$S^1 \subset \mathbb{C}$

$\overset{\shortparallel}{\{z \in \mathbb{C} \mid |z| = 1\}}$.

<u>Thm</u>: Let $p : E \to B$ is a covering map. If B_0 is a subspace of B and if $E_0 = p^{-1}(B_0)$ then $p_0 : E_0 \to B_0$ is a covering map.

↳ restricting p

<u>Thm</u>:- If $p : E \to B$
$p' : E' \to B'$ are covering maps

Then $p \times p' : E \times E' \longrightarrow B \times B'$

is a covering map.

$T^2 = S^1 \times S^1$

$\Rightarrow$ $\mathbb{R} \times \mathbb{R}$ is a covering space for the torus.

<u>Proof</u>:- Let $b \in B$, $b' \in B'$ and $U \ni b$, $U' \ni b'$ be evenly covered nbd by p and p' respectively.

Let $\{V_\alpha\}$ and $\{V_\alpha'\}$ be partitions of $p^{-1}(U)$ and

$(p')^{-1}(U')$ respectively.

Then $(p \times p')^{-1}(U \times U') = \bigsqcup V_\alpha \times V'_\beta$

and each $V_\alpha \times V_\beta$ is mapped homeomorphically onto $U \times U'$ by $p \times p'$. $\Rightarrow$ $E \times E'$ is a covering space of $B \times B'$.

$\boxtimes$

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