Lecture 10

- no problem session today.

Fundamental Groups

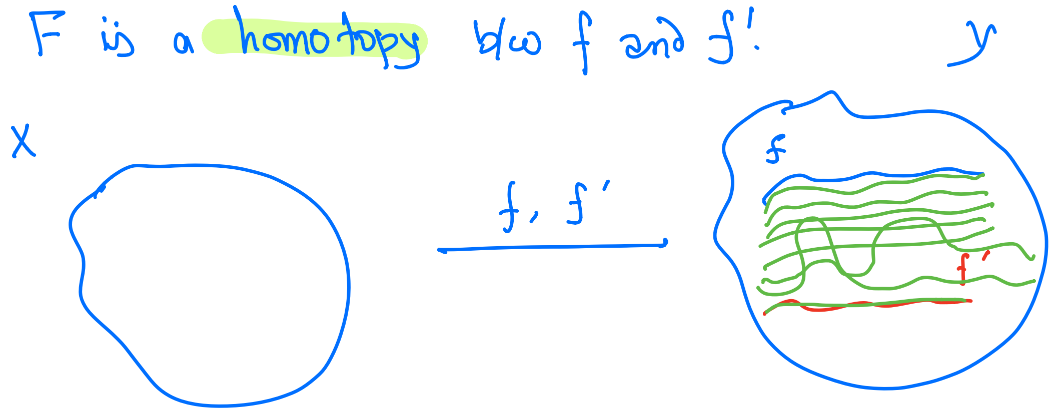
Homotopy of paths

Defn Let $f, f' : X \to Y$ be continuous maps.
f is homotopic to f' ($f \simeq f'$) is $\exists$ a continuous map $F : X \times I \to Y$ (here $I = [0,1]$) s.t.

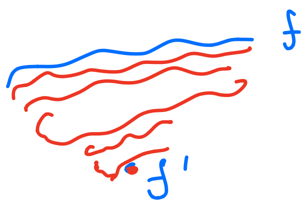
$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x)$$

$\forall\ x \in X$.

F is a homotopy b/w f and f'.



If f' is a constant map and $f \simeq f'$ then we say that f is null homotopic.



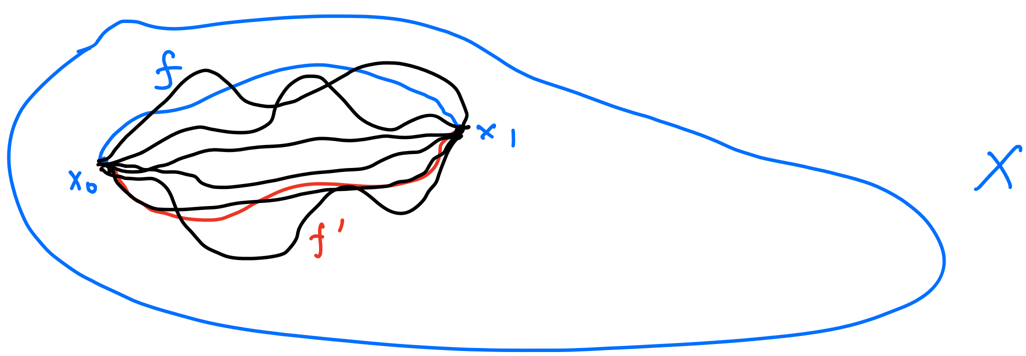
f is a path from x_0 to x_1 if $f : [0,1] \longrightarrow X$ continuous and $f(0) = x_0$
$f(1) = x_1$.

Def^n Two paths $f, f' : I \longrightarrow X$ are said to be path-homotopic if they have the same initial point x_0 and same final point x_1 and if $\exists$ continuous map $F : I \times I \longrightarrow X$ s.t.

$$F(s,0) = f(s) \quad \text{and} \quad F(s,1) = f'(s)$$

$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1$$

$\forall$ $s \in I$, $t \in I$. We write $f \simeq_p f'$.



Thm The relations $\simeq$ and $\simeq_p$ are equivalence relations.

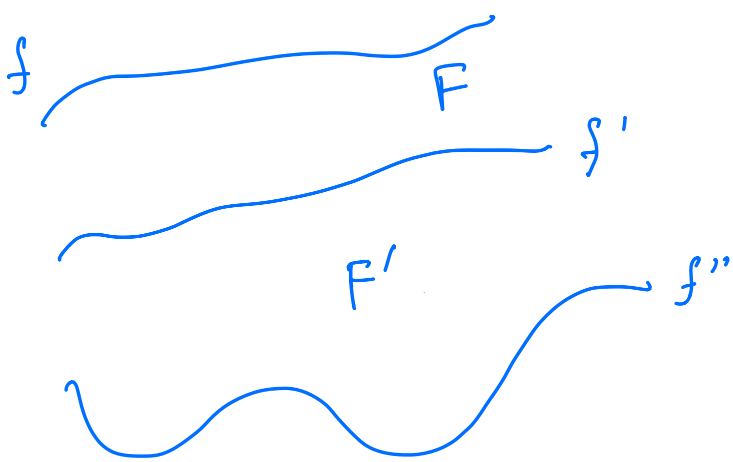
If f is a path then we'll denote its equivalence class $[f]$.

Proof:- $f \simeq f$, $F(x,t) = f(x)$

$f \simeq_p f$, $F(s,t) = f(s)$.

$f \simeq f' \Rightarrow f' \simeq f$. If F is the homotopy b/w f and f', then $G(x,t) = F(x, 1-t)$ is a homotopy b/w f and f'.

Let $f \simeq f'$ and $f' \simeq f''$. Want $f \simeq f''$.



Define $G : X \times I \longrightarrow Y$ as

$$G(x,t) = \begin{cases} F(x, 2t) & \text{for } t \in [0, 1/2] \\ F'(x, 2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

Ex. $f, g : X \longrightarrow \mathbb{R}^2$

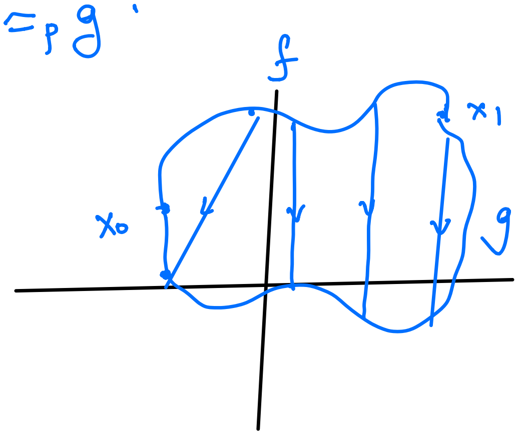
Then $f \simeq g$.

straight-line homotopy

$$F(x,t) = (1-t) f(x) + t g(x)$$

If f, g are paths in $\mathbb{R}^2$ from x_0 to x_1

then $f \simeq_p g$.

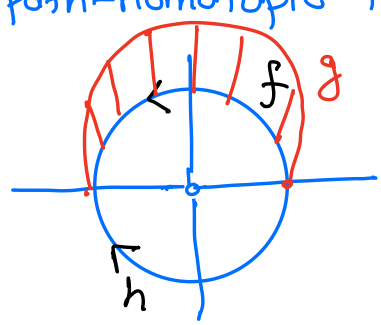


Let C be any convex subspace of $\mathbb{R}^n$

the straight line joining $a, b \in C$

lies in C.

Any two paths in C are path-homotopic to each other.



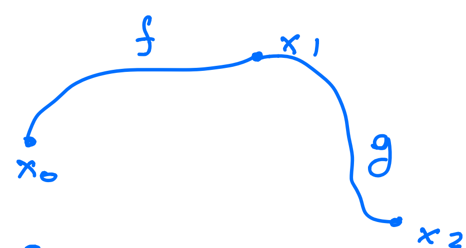
Ex. 2 $X = \mathbb{R}^2 \setminus \{0\}$

Consider

$f(s) = (\cos \pi s, \sin \pi s)$
$g(s) = (\cos \pi s, 2\sin \pi s)$
$h(s) = (\cos \pi s, -\sin \pi s)$

Product structure

Def f is path in X from x_0 to x_1 and let g is a path in X from x_1 to x_2.



We define the product

$f * g = h$ as

$$h(s) = \begin{cases} f(2s), & s \in [0, 1/2] \\ g(2s-1), & s \in [1/2, 1] \end{cases}$$

concatenation of f and g.

h is a path in X from x_0 to x_2.

$$[f] = \{ \gamma : [0,1] \to X, \gamma(0) = x_0, \gamma(1) = x_1 \text{ and } \gamma \simeq_p f \}$$

$$\boxed{[f] * [g] = [f * g]}$$

Exercise!

notice :- $[f] * [g]$ is not defined for every pair

- classes : we must have $f(1) = g(0)$.

Theorem :- The operation $*$ has the following properties:-

1) (Associative) If $[f] * ([g] * [h])$ is defined.
then so is $([f] * [g]) * [h]$ and

$$[f] * ([g] * [h]) = ([f] * [g]) * [h].$$

2. (Existence of right/left identities).
Given $x \in X$, let $e_x : I \longrightarrow X$ denote the constant path $e_x(s) = x$. If f is a path from x_0 to x_1 then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

3. (inverse) $\quad f : I \longrightarrow X$, $\quad f(0) = x_0$, $f(1) = x_1$

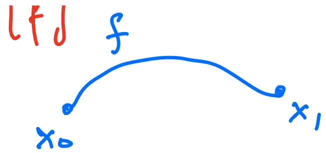
$$f^{-1}(s) = f(1-s) : I \longrightarrow X \qquad f^{-1}(0) = x_1, \quad f^{-1}(1) = x_0$$

reverse path of f. And

$$[f] * [f^{-1}] = [e_{x_0}] \quad \text{and} \quad [f^{-1}] * [f] = [e_{x_1}].$$

$\parallel$
$[f]^{-1}$

[f]



Proof :- $k: X \to Y$ continuous

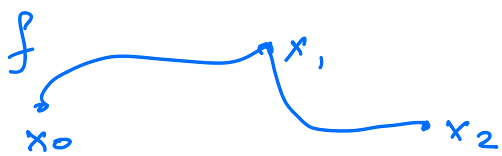
F is a path homotopy in X b/w f and f'

then $k \circ F$ is a path hom. in Y, b/w $k \circ f$, $k \circ f'$.

and

$f(1) = g(0)$ then

$$k \circ (f * g) = (k \circ f) * (k \circ g).$$



Let's look at 2) and 3).

e_0 is the constant path at $0 \in I$.

$i: I \to I$ identity map.

$e_0 * I$ is a path in I from 0 to 1.

$\because$ I is convex $\Rightarrow$ $\exists$ a path homotopy G in I b/w i and $e_0 * i$. Then

$f \circ G$ is a path homotopy in X b/w

$f \circ i = f$ and $f \circ (e_0 * i) = (f \circ e_0) * (f \circ i)$

$= e_{x_0} * f.$

$\Rightarrow$ we get (2).

(3). The reverse path of $i: I \to I$ is

$i^{-1}(s) = 1-s.$

Then $i * i^{-1}$ is a path in I beginning at 0 and is ending at 0. $\Rightarrow$ $i * i^{-1} \simeq_p e_0$. (I is convex)

Suppose H is the path hom. in I b/w e_0 and $i * i^{-1}$.

$\Rightarrow$ $f \circ H$ is a path homotopy b/w $f \circ e_0 = e_{x_0}$ and $(f \circ i) * (f \circ i^{-1}) = f * \underbrace{f^{-1}}$

reverse path as defined above.

define the "inverse" of $[f] = [f^{-1}]$.

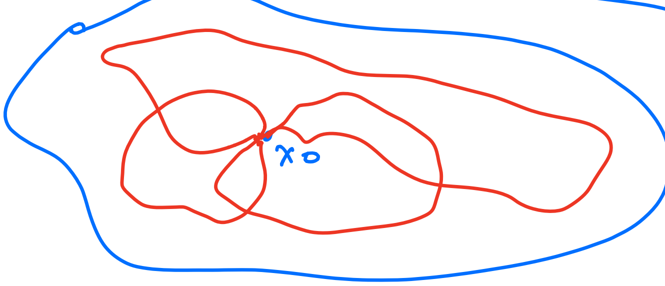
proof of (1) later. $\boxtimes$

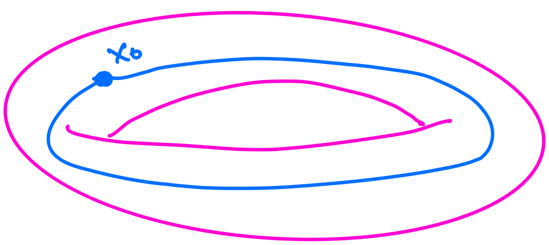
Pick a base point $x_0 \in X$. We are only going to look at loops in X at x_0, i.e., path in X which start at x_0 and end at x_0.

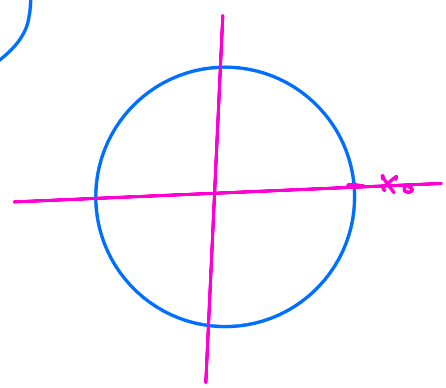
Def^n Let X be a space, $x_0 \in X$. The set of path homotopy classes of loops based at x_0 with the operation $*$ as above is a group called the fundamental group of X relative to the base point x_0. We denote it by $\pi_1(X, x_0)$.

$$\pi_1(X, x_0) = \{ [f] \mid f : I \to X \text{ w/ } f(0) = f(1) = x_0 \}.$$

first homotopy group.







Proof of associativity of $*$.

We describe $f*g$ in a different way.

Let $[a,b]$ and $[c,d]$ be two intervals in $\mathbb{R}$.

Then $\exists$ a unique $p:[a,b] \to [c,d]$

w/ $p(x) = mx+k$ s.t. $p(a) = c$ and $p(b) = d$.

Call p the positive linear map (plm)

of $[a,b] \to [c,d]$.

Now, consider $f*g$ as follows:- On $[0,1/2]$

$f*g = f \circ$ plm from $[0,1/2]$ to $[0,1]$

and on $[1/2,1]$

$f*g = g \circ$ plm from $[1/2,1] \to [0,1]$

Now let f, g and h be paths in X s.t.

$f*(g*h)$ and $(f*g)*h$ are defined, i.e.,

$$f(1) = g(0) \text{ and } g(1) = h(0).$$

Define a "triple" product of the paths f, g and h as follows:-

Let $a, b \in I$ w/ $0 < a < b < 1$. Define a path $k_{a,b}$ in X as

$$k_{a,b} = \begin{cases} f \circ \text{plm from } [0,a] \text{ to } [0,1] & \text{on } [0,a] \\ g \circ \text{plm from } [a,b] \text{ to } [0,1] & \text{on } [a,b] \\ h \circ \text{plm from } [b,1] \text{ to } [0,1] & \text{on } [b,1] \end{cases}$$

<u>Claim</u> :- If $c, d \in I$ w/ $0 < c < d < 1$ are another pair of points then $k_{a,b} \simeq_p k_{c,d}$.

Notice that if we prove the claim then

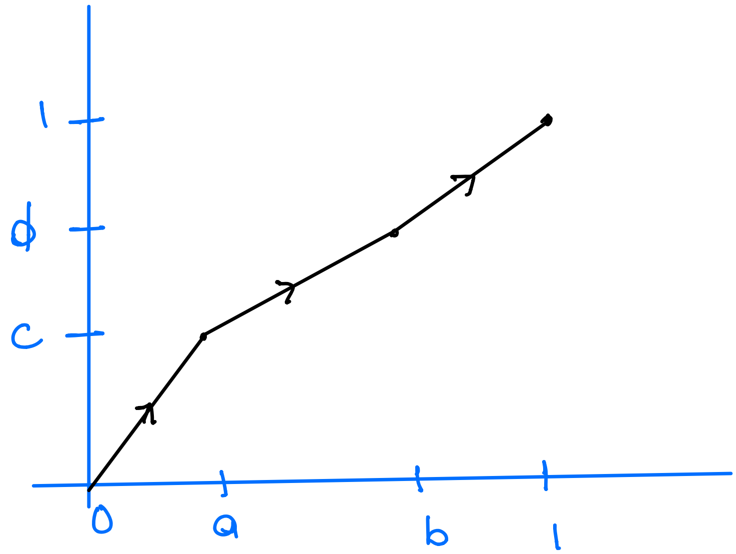
$$f * (g * h) = k_{a,b} \quad \text{w/ } a = 1/2,\ b = 3/4$$

and $(f * g) * h = k_{c,d}$ w/ $c = 1/4$, $d = 1/2$

$$\Rightarrow [f] * ([g] * [h]) = ([f] * [g]) * [h].$$

<u>Proof of the Claim</u> :-

Let $p : I \longrightarrow I$ be the map as described below.



i.e., $p|_{[0,a]}$ = plm from $[0,a]$ to $[0,c]$

$p|_{[a,b]}$ = plm from $[a,b]$ to $[c,d]$

$p|_{[b,1]}$ = plm from $[b,1]$ to $[d,1]$.

Then $k_{c,d} \circ p = k_{a,b}$

Now $\because$ $p: I \to I$ and I is convex $\Rightarrow$

$\exists$ a path homotopy P in I b/w p and $i: I \to I$.

Then $k_{c,d} \circ P$ is a path homotopy in X b/w $k_{a,b}$ and $k_{c,d}$. $\blacksquare$