# Problem Session

① $G$ group $\quad [G,G] = \langle [x,y] = xyx^{-1}y^{-1} \mid x,y \in G \rangle$

If $[x,y] = xyx^{-1}y^{-1} = e \implies xy = yx.$

$[G,G] = e \iff G$ is abelian.

ⓐ $[G,G] \trianglelefteq G.$

$gng^{-1} \in [G,G]$, $\forall\, n \in [G,G]$

$n = x_1y_1x_1^{-1}y_1^{-1}\, x_2y_2x_2^{-1}x_2^{-1}, \ldots\, x_ny_nx_n^{-1}y_n^{-1}$

$[x,y] = xyx^{-1}y^{-1}$

$g\cdot[x,y]g^{-1} = aba^{-1}b^{-1}$ for some $a,b \in G.$

$\quad\quad\quad\quad\quad = [a,b]$

$\quad ||$

$$gxyx^{-1}y^{-1}g^{-1} = \underbrace{gxg^{-1}}_{gx^{-1}g^{-1}}gyx^{-1}y^{-1}g^{-1}$$

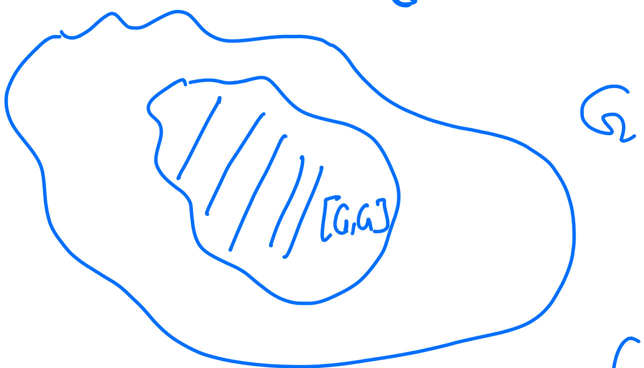
$$= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1}$$

$$= \underbrace{(gxg^{-1})}_{a}\underbrace{(gyg^{-1})}_{b}\underbrace{(gxg^{-1})^{-1}}_{a^{-1}}\underbrace{(gyg^{-1})^{-1}}_{b^{-1}}$$

$$= [a,b]$$

$\implies g[x,y]g^{-1} \in [G,G] \implies [G,G] \trianglelefteq G.$

(b) $G^{ab} = G/\underbrace{[G,G]}_{C}$ abelianization of $G$.




$(aba^{-1}b^{-1} = e)$
$\Longleftrightarrow ab = ba$

$xC, yC \in G/[G,G]$

$(xC)(yC) = (yC)(xC)$
want.

$(xC)(yC) = xyC$

$\underbrace{xyx^{-1}y^{-1}}_{[x,y]}C = C \implies xyC = yxC$

$\implies G/[G,G]$ is abelian.

if $G$ is abelian then a) $\implies [G,G] = e$

$\implies G^{ab} = G/[G,G] = G$.

$\pi_1(X)^{ab} = H_1(X)$ = 1st homology group

$\pi_n(X)^{ab} = H_n(X)$ = n-th homology group.

(c) $(\text{free product of groups})^{ab}$ = free abelian groups

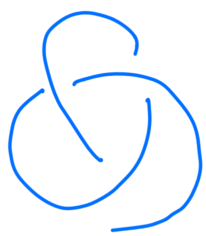
$(\text{free abelian group of rank } n)^{ab} \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\text{-times}}$

$\langle a_1 \rangle, \langle a_2 \rangle \ldots \langle a_n \rangle$

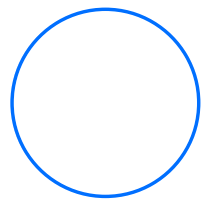
(d) $G = \{x, y \mid x^2 = y^3\} \cong \pi_1(\mathbb{R}^3 \setminus K_{2,3})$

$K_{2,3}$ = trefoil knot.

$G^{ab} \cong \mathbb{Z}$



$K_{2,3}$



$K_{1,0}$

$\pi_1(\mathbb{R}^3 \setminus K_{p,q}) \cong \{x, y \mid x^p = y^q\}$

$\pi_1(\mathbb{R}^3 \setminus K_{1,0}) = \{x, y \mid x^1 = y^0\}$

$\Uparrow$

$x = e$

<u>Fact</u> :- $G^{ab}$ for any knot group $G \cong \mathbb{Z}$.

$G = \{x, y \mid x^2 = y^3\}$

$G^{ab} \cong \mathbb{Z}$

$\varphi : G \longrightarrow \mathbb{Z}$

$\varphi(x), \ \varphi(y) \in \mathbb{Z}$

$x^2 = y^3 \Rightarrow \varphi(x^2) = \varphi(y^3)$

$G = \{x, y \mid x^2 = y^3\}$ $\quad x \cdot x = y \cdot y \cdot y$

<u>Vaughan Jones</u>

Fields medal 1990

$[x, y] = x y x^{-1} y^{-1}$

$\langle x, y \mid x^2 = y^3 \rangle \cong \langle a, b \mid bab = aba \rangle$

"="

Braid group

$$bab = aba \rightsquigarrow x^2 = y^3$$

$$a = xy^{-1}, \quad b = y^2x^{-1}$$

$$G^{ab} = G/[G,G] = \langle a,b \mid bab = aba,\ [a,b]=e \rangle$$

$$ab = ba$$

$$bab = aba$$

$$bab = aba = a \cdot ab = a^2 b$$

$$\Rightarrow \quad ba = a^2 \Longleftrightarrow b = a$$

$$G/[G,G] = \langle a,b \mid bab = aba,\ [a,b]=e \rangle$$

$$= \langle a,b \mid a = b \rangle$$

$$= \langle a \rangle \cong \mathbb{Z}$$

$$\therefore \quad G^{ab} \cong \mathbb{Z}.$$

② $\Sigma_g$

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, a_2 b_2, \ldots, a_g, b_g \mid [a_1,b_1]\cdot[a_2,b_2] \cdots [a_g,b_g] = e \rangle$$

$$\frac{\pi_1(\Sigma_g)}{[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]} \cong \frac{F_{2g}/N}{[F_{2g}/N, F_{2g}/N]} \cong F_{2g}/[F_{2g}, F_{2g}]$$

$$\therefore \quad \pi_1(\Sigma_g)^{ab} \cong (F_{2g})^{ab}$$

free group on 2g gen.

$$\cong \mathbb{Z}^{2g} \quad (1.c)$$

$$= \underbrace{\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}}_{2g}$$

$\Sigma_g \not\cong \Sigma_h$ if $g \neq h$

if $\Sigma_g \not\cong \Sigma_n \implies \pi_1(\Sigma_g) \cong \pi_1(\Sigma_h)$

$$\Downarrow$$

$$\pi_1(\Sigma_g)^{ab} \cong \pi_1(\Sigma_h)^{ab}$$

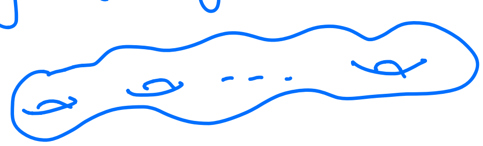
$$\Downarrow$$

$$\mathbb{Z}^{2g} \cong \mathbb{Z}^{2h}$$

only possible iff $g = h$

$\therefore \quad \Sigma_g \not\cong \Sigma_h.$

$S^2$, $\mathbb{RP}^2$, $\Sigma_g$ for genus $g$. Uniformisation theorem.



③ For $w \in \mathbb{R}^n$, a ray $R$ from $w = \{ w + tp \mid p \in \mathbb{R}^n \setminus \{0\}$ is fixed, $t \in \mathbb{R}_{\geq 0} \}$

we prove something more here.

If $U$ is a bounded convex set in $\mathbb{R}^n$ then we prove that $\exists$ a homeomorphism of $\bar{U}$ w/ $B^n$ carrying $\partial U$ to $S^{n-1}$.

Suppose $w = 0$. The eq. $f(x) = \frac{x}{\|x\|}$ gives a continuous map from $\mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$.

Now, each ray $R$ emanating from $w \in U$ intersects $\partial U = \bar{U} - U$ in precisely one point. $\Rightarrow$ $f_{|\partial U}$ is a bijection $\Rightarrow$ it is a homeomorphism w/ $S^{n-1}$ and $\because$ $\partial U$ is compact $\Rightarrow$

Consider $f^{-1} : S^{n-1} \longrightarrow \partial U$. Extend it to a bijection, say $F : B^n \longrightarrow \bar{U}$ by letting $F$ map the line segment joining $0$ to some point $u \in S^{n-1}$ linearly onto the line ~~segment~~ joining $0$ to $f^{-1}(u)$, i.e.,

$$F(x) = \begin{cases} \|f^{-1}(x/\|x\|)\| \, x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For $x \neq 0$, $F$ is clearly continuous. For $x = 0$: $\because$ $f^{-1}(x)$ is an element of $U$ $\Rightarrow$ $\exists$ $M$ s.t. $\|f^{-1}(x)\| < M$ $\forall x \in S^{n-1}$

$\therefore$ whenever $\|x - 0\| < \delta$ $\Rightarrow$

$\| F(x) - F(0) \| = \| \|f^{-1}(x/\|x\|)\| x - 0 \| \leq M\delta$ $\Rightarrow$ $G$ is cont. at $0$ as well. $\boxed{\text{ls}}$

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