Problem Session 6

Prob. set 5

① $h : S^1 \to S^1$ is nullhomotopic.

want:- $h(x) = x$ for some $x \in S^1$.

from the theorem in lec. h extends to a continuous map $k : B^2 \to S^1$ $(k|_{S^1} = h)$

$$\begin{array}{ccc} k : B^2 & \longrightarrow & S^1 \\ & & \downarrow j \\ & & B^2 \end{array}$$

$j \circ k : B^2 \to B^2$ continuous.

$\therefore$ by Brouwer's f.p.t. $\exists\ x \in B^2$ s.t

$j(k(x)) = x \implies \underset{S^1}{\overset{\cap}{k(x)}} = x \implies x \in S^1$

$\therefore$ $x \in S^1$ w/ $k(x) = x$
$\quad\quad\quad\quad\quad\quad \| $
$\quad\quad\quad\quad\quad\quad h(x) = x.$

Want : $\exists\ x \in S^1$ s.t. $h(x) = -x$.

$\alpha : S^1 \to S^1$, i.e. $\alpha(x) = -x$.

$\because$ h is nullhomotopic $\implies$ $\exists$ $x \in S^1$ s.t.

$h \simeq e_x \implies$ homotopy $H : S^1 \times I \to S^1$

s.t $H(s,0) = h(s)$ and $H(s,1) = e_x(s) = x$.

$\implies$ $\alpha \circ H$ is a hom. b/w $\alpha \circ h$ and $\alpha \circ e_x = -x$

$\Rightarrow \alpha \circ h \simeq e_{-x}$ $\quad \alpha \circ h : S^1 \to S^1$ is nullhomotopic

$\Rightarrow \exists$ a fixed point $\alpha \circ h \Rightarrow \exists\, y \in S^1$ s.t

$\alpha(h(y)) = y \Rightarrow -h(y) = y \Rightarrow h(y) = -y.$

$\therefore$ y is the required point. $\boxtimes$

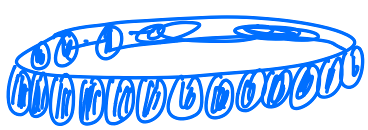
(2) $h : S^n \to S^m$ antipode-preserving if
$h(-x) = -h(x)$ $\forall x \in S^n$.

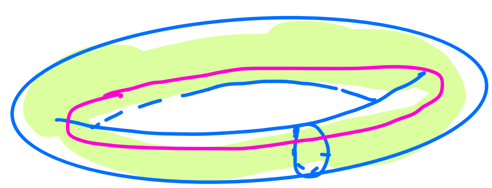
If $h : S^1 \to S^1$ is cont. and antipode-preserving then h is not nullhomotopic.

$\nexists$ $g : S^2 \to S^1$, g antipode-preserving.

Borsuk-Ulam thm $f : S^2 \to \mathbb{R}^2$ $\exists\, x \in S^2$ s.t. $f(x) = f(-x)$.

(3) (a) solid torus $B^2 \times S^1$





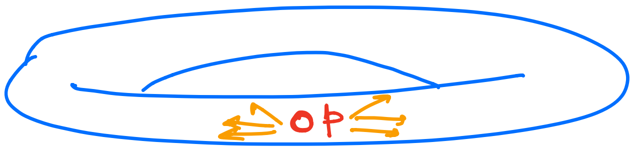
defor. retracts to S^1

$\therefore \pi_1(B^2 \times S^1, x_0) \cong (\mathbb{Z}, +)$.

(b) $T^2 - \{p\}$

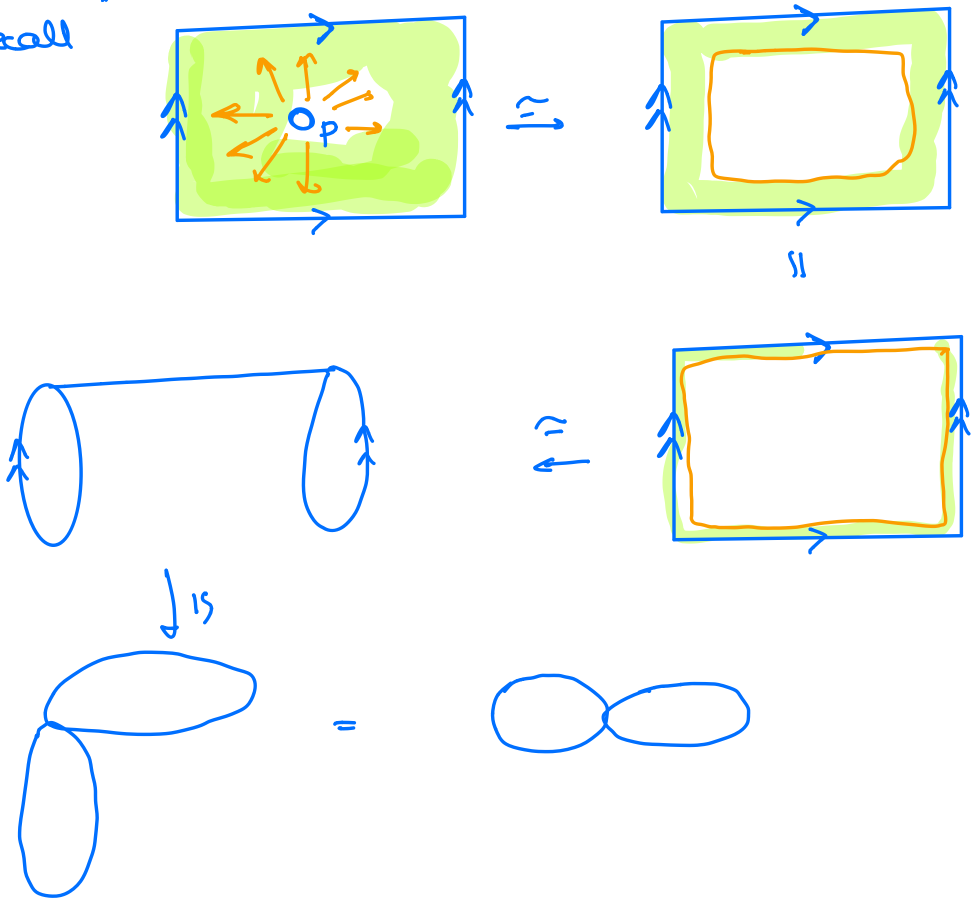
$\pi_1(T^2 - \{p\}, x_0) \cong \pi_1(8)$

non-abelian



$\mathbb{Z} \times \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$ (abelian)

Recall $T^2 =$

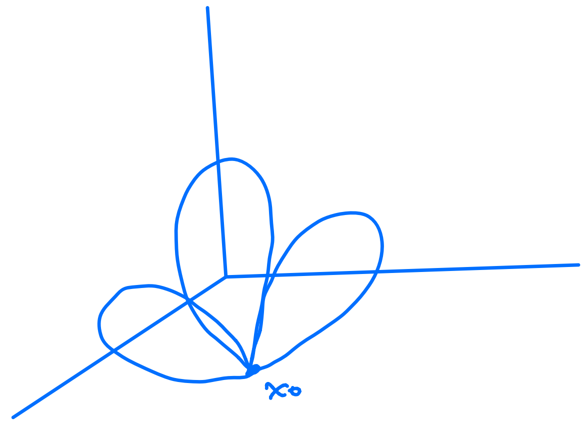


(c) cylinder $S^1 \times \mathbb{R} \simeq \mathbb{Z}$.

d) $\mathbb{R}^3 \setminus$ non-negative x, y, z axes.

$\mathbb{R}^2 - \{p\} - \{q\} \sim \infty$ or 

X



$\cap$

$\mathbb{R}^3 - \{0\} \longrightarrow S^2 \subset \mathbb{R}^3$

$x \longmapsto \dfrac{x}{\|x\|}$

homotopy b/w X and $S^2 - \{(1,0,0), (0,1,0), (0,0,1)\}$

stereographic projection

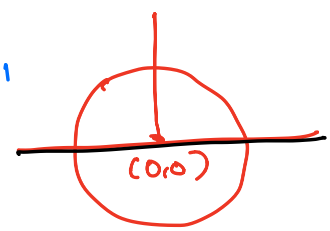
$S^2 - \{p\} - \{q\} - \{r\} \simeq \mathbb{R}^2 - \{q\} - \{r\}$

$\wr\mid$

$\pi_1(8) \cong \pi_1(\Theta)$.

(e) $\{x \mid \|x\| > 1\} \subset \mathbb{R}^2$

(f) $S^1 \cup (\mathbb{R}_+ \times 0) \subset \mathbb{R}^2 \simeq S^1$



$S^1 \cup (\mathbb{R} \times 0) \subset \mathbb{R}^2 \wr\mid$

$\wr\mid$ Θ space

5

$h: S^1 \to S^1$ cont.

$h_*: \pi_1(S^1, x_0) \to \pi_1(S^1, x_1)$

$h_*(\gamma(x_0)) = d \cdot \gamma(x_1)$

$\overset{\shortparallel}{\text{degree of } h.}$

$h: S^n \to S^n$

$h_*: H_n(S^n) \to H_n(S^n)$

n-th homology

$H_n(S^n) \cong \mathbb{Z}$, $H_n(S^n) \cong \mathbb{Z}$

5 (a) This is a cleaner version of what I wrote during the Problem session.

Let $h: S^1 \to S^1$ w/ $h(x_0) = x_1 \in S^1$

Suppose $y_0 \in S^1$ and let β be a path from x_0 to y_0. Let $y_1 = h(y_0)$. Then $\beta' = h \circ \beta$ is a path from x_1 to y_1.

$$\therefore \quad \gamma_{y_1} = [\beta']^{-1} * \gamma_{x_1} * [\beta]'$$

$$\therefore \quad h_*(\gamma_{y_0}) = h_*\left([\beta]^{-1} * \gamma_{x_0} * [\beta]\right)$$

$$= [\beta']^{-1} * h_*(\gamma_{x_0}) * [\beta'] \quad (\because \beta' = h \circ \beta)$$

$$= [\beta']^{-1} * d \cdot \gamma_{x_1} * [\beta']$$

$$= d\left([\beta']^{-1} * \gamma_{x_1} * [\beta']\right)$$
$$= d \cdot \gamma_{y_1}$$

$\therefore$ the degree is independent of the choice of x_0.

(b) This just follows from the lemma which we proved in the lecture that if $h \simeq k$ w/ $h(x_0) = k(x_0)$, then $h_* = k_*$.

This is an important fact and we will use this frequently in applications later.

d) $\because$ the constant map maps any loop in S^1 to the constant loop e_c $\Rightarrow$ the generator

$$[\gamma_{x_0}] \in \pi_1(S^1, 0) \longmapsto [e_c] \Rightarrow h_*([\gamma(x_0)]) = 0 \cdot \gamma(x_1)$$

$\Rightarrow \deg = 0$.

the identity map induces the identity homomorphism $\Rightarrow$ $id_*(\gamma(x_0)) = 1 \cdot \gamma(x_0)$

$\Rightarrow \deg(id) = 1$.

for $\rho(x_1, x_2) = (x_1, -x_2)$

if we consider the standard covering map

$p : \mathbb{R} \to S^1$ by $p(x) = (\cos 2\pi x, \sin 2\pi x)$

then the generator $[\gamma]$ is $\gamma(x) = (\cos 2\pi x, \sin 2\pi x)$

$\therefore \quad (\rho \circ \gamma)(x) = (\cos 2\pi x, -\sin 2\pi x)$

$= \gamma(1-x) = \gamma^{-1}(x)$

$\therefore$ under ρ_*, the generator $[\gamma]$ is mapped to

$[\gamma]^{-1} \implies d = -1.$

$h(z) = z^n.$

here we are $S^1 \subset \mathbb{C}$ w/ $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

Then following the same notation as above,

the generator $\gamma(x) = e^{2\pi i x}$.

$\implies \quad h \circ \gamma(x) = e^{2\pi i n x}$

$= \underbrace{e^{2\pi i x} \cdot e^{2\pi i x} \cdots e^{2\pi i x}}_{n\text{-times}}$

$\therefore$ the generator under the homomorphism h_* is wrapped around the circle n times.

$\Longrightarrow \deg h = n.$ $\blacksquare$