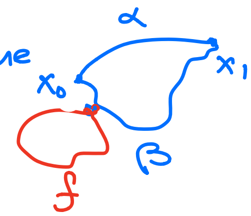
Problem Session 6

Problem Set 4

(1) $x_0, x_1 \in X$ path connected.

$$\hat{\alpha}([f]) = [\alpha]^{-1} * [f] * [\alpha]$$ where



Suppose $\pi_1(X, x_0)$ is abelian.

Want:- $\hat{\alpha} = \hat{\beta}$, $[f] \in \pi_1(X, x_0)$

$$\hat{\alpha}([f]) = [\alpha]^{-1} * [f] * [\alpha]$$
$$= [\alpha]^{-1} * [f] * [\beta] * [\beta]^{-1} * [\alpha]$$
$$= \left([\alpha]^{-1} * [f] * [\beta]\right) * \left([\beta]^{-1} * [\alpha]\right)$$
$$= [\beta]^{-1} * [\alpha] * [\alpha]^{-1} * [f] * [\beta]$$
$$= [\beta]^{-1} * [f] * [\beta] = \hat{\beta}([f])$$

for the other direction , $[f], [g] \in \pi_1(X, x_0)$

$$[f] * [g] = [g] * [f] \quad \text{(Want)}$$

and α is a path b/w x_0 and x_1 then $f * \alpha$ is again a path b/w x_0 and x_1.

$$\therefore \quad \widehat{f * \alpha}([g]) = \hat{\alpha}([g])$$

$\Rightarrow \quad [f*\alpha]^{-1} * [g] * [f*\alpha] = [\alpha]^{-1} * [g] * [\alpha]$

$\Rightarrow \quad [\alpha]^{-1} * [f]^{-1} * [g] * [f] * [\alpha] = [\alpha]^{-1} * [g] * [\alpha]$

$\Rightarrow \quad [f]^{-1} * [g] * [f] = [g]$

$\Rightarrow \quad [g] * [f] = [f] * [g] \Rightarrow \pi_1(X, x_0)$ is abelian.

$\boxed{\text{ii}}$

② $\pi_1(X \times Y, (p,q)) \cong \pi_1(X, p) \times \pi_1(Y, q)$

$\varphi_1 : X \times Y \longrightarrow X$, $(x,y) \longmapsto x$

$\varphi_2 : X \times Y \longrightarrow Y$, $(x,y) \longmapsto y$

③ $A \subset X$ $\quad r : X \longrightarrow A$ s.t. $r|_A = id_A$.

a) $a_0 \in A$ then $r_* : \pi_1(X, a_0) \longrightarrow \pi_1(A, a_0)$ is surjective.

b) $i : A \hookrightarrow X \rightsquigarrow i_* : \pi_1(A) \hookrightarrow \pi_1(X)$ is injective.

$r \circ i : A \longrightarrow A$ is the identity map of A. $\quad r \circ i = id_A$

$\Rightarrow \quad (r \circ i)_* = (id_A)_* \Rightarrow r_* \circ i_* = id_*$ of $\pi_1(A, a_0)$.

If $[f] \in \pi_1(A, a_0)$ then $r_*(i_*([f])) = [f]$

$\therefore$ r_* is surjective.

i_* is injective.

(c) If S^1 were a retract of B^2
then $i: S^1 \longrightarrow B^2$ would be injective (part (b))

$i_*: \underbrace{\pi_1(S^1, b_0)}_{\mathbb{Z}} = \underbrace{\pi_1(B^2, b_0)}_{0}$

NOT possible

$\Longrightarrow$ no retraction from B^2 to S^1 ☑

(4) $p: E \longrightarrow B$ covering map, B is connected.

If $\exists\ b_0 \in B$ s.t. $p^{-1}(b_0)$ has k-elements
then $p^{-1}(b)$ has k elements for any $b \in B$.

Use connectedness of B.

$A = \{ b \in B \mid p^{-1}(b)$ has k elements $\}$

$\neq \phi$ as $b_0 \in A$.

All we need to show is that A is open. (Sim. prove that
A^c is open $\Longrightarrow$ $A = B$ as B is connected).

A is open:
Let $a \in A$ $\Rightarrow$ $p^{-1}(a)$ has k elements
$\because$ p is a covering map. $\Rightarrow$ $\exists$ $U \ni a \subset B$ open

s.t. $\quad p^{-1}(U) = \underbrace{\bigsqcup_{\alpha \in I} V_\alpha}_{\text{disjoint.}} \qquad p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

I must have k-elements.

$\Rightarrow \quad p^{-1}(U) = V_1 \cup V_2 \cup \dots \cup V_k.$

Want that $\quad U \subset A \quad$, i.e., if $\; x \in U \;$ then we must show that $\; p^{-1}(x) \;$ must have k elements.

$p^{-1}(x) \subseteq p^{-1}(U) = \bigsqcup_{i=1}^{k} V_i$

$\Rightarrow \quad p^{-1}(x)$ has k elements $\Rightarrow \quad x \in A$

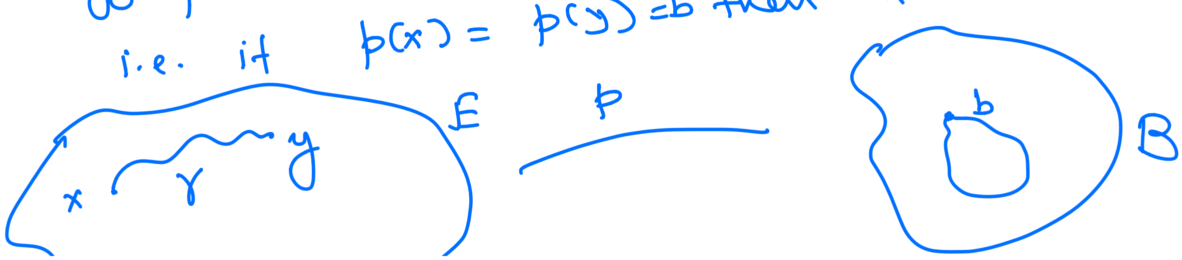
$\Rightarrow \quad U \subseteq A \quad \Rightarrow \quad A$ is open.

(5) $\quad p: E \to B$ covering map, $\; E$ path-connected.
If B is simply-connected $\quad p$ is a homeomorphism.

p is continuous, surjective.

p is an open map, i.e., $\; p$ takes open sets in E to open sets in B.

To prove p is a homeomorphism, $\; p$ is injective,
i.e. if $\quad p(x) = p(y) = b$ then $\quad x = y$.



$\because$ B is simply-connected

$\Rightarrow [p \circ \gamma] = [e_b]$

by the path-lifting lemma, we get $x = y$

$\Rightarrow$ p is a homeomorphism.

<u>p is an open map.</u>

$A \underset{\text{open}}{\subset} E$ $\quad p(A) \underset{\text{open}}{\subset} B$ $\quad$ Let $x \in p(A)$.

Let $U \ni x$ be an evenly covered nbd of x.

$$p^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha$$

$\exists \; y \in A$ s.t. $p(y) = x$

Suppose $V_\beta \ni y$ $\quad V_\beta \cap A$ is open in E

$V_\beta \cap A$ is open in V_β

$\because$ $p : V_\beta \to U$ is a homeomorphism

$p(V_\beta \cap A) \subset U$ is open.

$\Rightarrow$ $p(V_\beta \cap A)$ is open in B

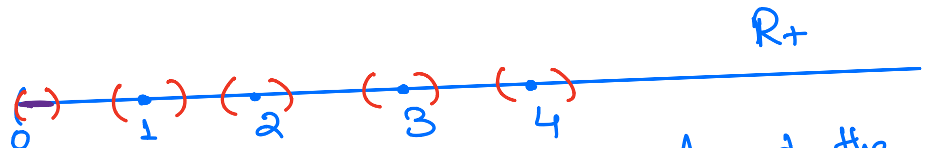
$\therefore$ $p(V_\beta \cap A) \ni x$ and $p(V_\beta \cap A) \subset p(A)$

$\Rightarrow$ $p(A)$ is open.

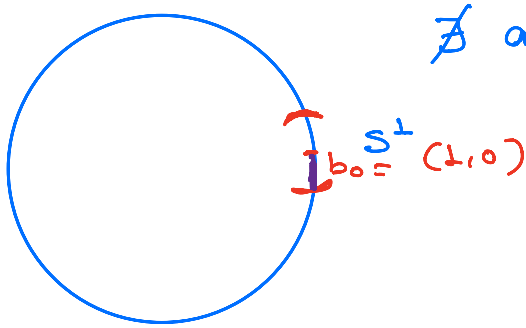
$\boxed{\checkmark}$

⑥ $p : \mathbb{R}_+ \longrightarrow S^1$

$p(x) = (\cos 2\pi x, \sin 2\pi x)$



Around the point $b_0 \in S^1$ $\nexists$ any evenly covered nbd $\Rightarrow$ $p|_{\mathbb{R}_+}$ is not a covering map.



<u>Local homeo.</u>

$e \in E$, $p(e) = x \in B$.

$\because$ p is a covering map $\Rightarrow$ $\exists$ $U \ni x$ $\subseteq_{open}$ B

$\Rightarrow$ $p^{-1}(U) = \bigsqcup V_\alpha$, $p : V_\alpha \to U$ homeomorphism.

$\because$ $p(e) = x \in U \Rightarrow e \in p^{-1}(U) = \bigsqcup V_\alpha$

$\Rightarrow$ e is in V_β for $\beta \in I$

$\Rightarrow$ $p|_{V_\beta} : V_\beta \to U$ is the required local homeomorphism. $\boxtimes$