# Problem Session 4

**Tychonoff's Theorem** If $\{X_\alpha\}_{\alpha\in I}$ is a collection of top. spaces w/ $X_\alpha$ is compact $\forall\, \alpha\in I$ then

$$X = \prod_{\alpha\in I} X_\alpha$$

is compact in the product topology.

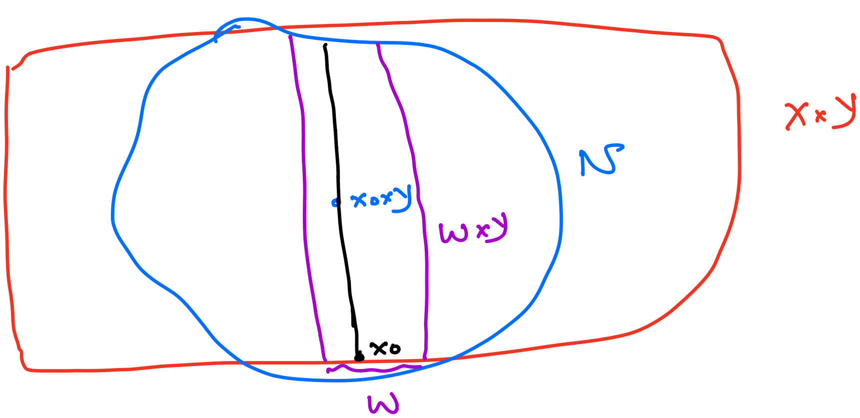
**Remark:-** $X$ is not nec. compact in the box topology.

**Finite product of compact spaces is compact**

If $X$ and $Y$ are compact then so is $X\times Y$.

**Step 1** Suppose $X$ and $Y$ are top. spaces w/ $Y$ compact. Let $x_0 \in X$. Consider the slice $x_0\times Y$ in $X\times Y$ which contains $x_0\times Y$ and let $N$ be an open set in $X\times Y$ which contains $x_0\times Y$. Then

$\exists$ a nbd $W$ of $x_0$ in $X$ s.t. $\underbrace{W\times Y}_{\text{tube about } x_0\times Y} \subset N$.



Let's cover $x_0 \times Y$ by basis elements $U \times V$ in $X \times Y$.
w/ $U \times V \subset N$.

$\because$ $x_0 \times Y \simeq Y$ which is compact

$\Rightarrow$ there are finitely many basis elements $U_1 \times V_1, U_2 \times V_2, \dots, U_n \times V_n$ s.t. $x_0 \times Y \in U_i \times V_i$ and $U_i \times V_i \subset N$. Define

$$W = U_1 \cap U_2 \cap \dots \cap U_n$$ – open set contains $x_0$.

Note that the sets $U_i \times V_i$ which cover $x_0 \times Y$ actually cover $W \times Y$. The reason is :-

$$x \times y \in W \times Y$$

$x_0 \times y \in U_i \times V_i$ for some $i$ $\Rightarrow$ $y \in V_i$

$\because$ $x \in U_j$ $\forall j$ $(x \in W)$ $\Rightarrow$ $x \times y \in U_i \times V_i$

$\Rightarrow$ $\{U_i \times V_i\}_{i=1}^{n}$ indeed cover $W \times Y$.

$\because$ $U_i \times V_i \subset N$ $\Rightarrow$ $W \times Y \subset N$.

<u>Step 2</u>. $X$ and $Y$ are compact. Suppose $\mathcal{A}$ is an open cover of $X \times Y$.

Let $x_0 \in X$ $\Rightarrow$ $x_0 \times Y$ is compact $\Rightarrow$ $\exists$ finitely many elements $A_1, \dots, A_m$ of $\mathcal{A}$. which covers $x_0 \times Y$.

$N = A_1 \cup A_2 \cup \dots \cup A_m$ is an open set in $X \times Y$

containing $x_0 \times Y$ $\Rightarrow$ by step 1 $\exists$ a tube $W \times Y$ about $x_0 \times Y$ ($W \underset{\text{open}}{\subset} \text{in } X$).

$W \times Y$ is covered by finitely many elements $A_1, A_2, \ldots A_m$ of $\mathcal{A}$.

$\therefore$ $\forall x \in X$ $\exists$ $W_x \ni x$, $W_x \underset{\text{open}}{\subset} X$ s.t. the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$.

$\{W_x\}_{x \in X}$ is an open cover for $X$ $\Rightarrow$ $\because$ $X$ is compact $\exists$ finite subcollection $\{W_1, W_2, \ldots, W_k\}$ covering $X$.

$$X \times Y \subset \underbrace{W_1 \times Y}_{\substack{\text{finitely}\\ \text{many}\\ \text{elements}\\ \text{of } \mathcal{A}}} \cup \underbrace{W_2 \times Y} \cup \ldots \underbrace{W_k \times Y}$$

$\Rightarrow$ $\mathcal{A}$ admits a finite subcover.

$\Rightarrow$ $X \times Y$ is compact. $\square$

<u>**Def$^n$**</u> Let $X$ be a set. A collection $\mathcal{C}$ of subsets of $X$ is said to have the finite intersection property if $\forall$ finite subcollection $\{C_1, C_2, \ldots, C_m\}$ of $\mathcal{C}$

$$C_1 \cap C_2 \cap \ldots \cap C_m \neq \phi.$$

**Theorem** Let $X$ be a top. space. Then $X$ is compact $\Longleftrightarrow$ for every collection $\mathcal{C}$ of closed sets in $X$ having the f.i.p., $\bigcap_{C \in \mathcal{C}} C \neq \phi$.

Proof: If $\mathcal{A}$ is a collection of subsets in $X$ then $\mathcal{C} = \{ X \setminus A \mid A \in \mathcal{A} \}$ satisfies:-

1) $\mathcal{A}$ is a collection of open sets $\Longleftrightarrow$ $\mathcal{C}$ is collection of closed sets.

2) $\mathcal{A}$ covers $X$ $\Longleftrightarrow$ $\bigcap_{C \in \mathcal{C}} C = \phi$.

$$X \subset \bigcup_{A \in \mathcal{A}} A \Longleftrightarrow \bigcap_{A \in \mathcal{A}} A^c = \phi \Longleftrightarrow \bigcap_{C \in \mathcal{C}} C = \phi.$$

3) A finite subcollection $\{A_1, \dots, A_n\}$ of $\mathcal{A}$ covers $X$ $\Longleftrightarrow$ $\bigcap_{i=1}^{n} C_i = \bigcap_{i=1}^{n} X \setminus A_i = \phi$.

The proof the theorem follows by looking at the contrapositive statement and by the 3 properties above. ▨

**Counterexample for the remark above**

$$\underbrace{[0,1] \times [0,1] \times [0,1] \times \cdots [0,1] \times \cdots}_{\text{countable product}}.$$

If $X = \prod_{k \in \mathbb{N}} [0,1]_k$ were compact in the box top.

$\Rightarrow$ the closed subset $Y = \prod_{k \in \mathbb{N}} \{0,1\}_k$ of $X$ must be compact.

$[0,1) \times [0,1) \times \cdots \times [0,1) \times \cdots$ open in $X$ in box topology.

$Y$ is an infinite set, closed and is discrete.

$\Rightarrow$ $Y$ can never be compact.

$\therefore$ $X$ is not compact.

## Proof of the Tychonoff's Theorem

**Lemma A** Let $X$ be a set and let $\mathcal{A}$ be a collection of subsets of $X$ having the f.i.p. Then $\exists$ a collection $\mathcal{D}$ of subsets of $X$ s.t. $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{D}$ has f.i.p and no collection of subsets of $X$ satisfying fip contains $\mathcal{D}$ properly, i.e.

$\mathcal{D}$ is the maximal collection of subsets of $X$ satisfying the f.i.p.

**Zorn's Lemma** A partially ordered set $P$ having the property that every chain in $P$ has an upper bound in $P$, contains at least one maximal element.

$P$ having a partial order. $\leq$ s.t. If $a, b, c \in P$

$a \leq a$

$a \leq b,\ b \leq a \Rightarrow a = b$

$a \leq b,\ b \leq c \Rightarrow a \leq c.$

— A chain $C$ in $P$ is a subset of $P$ which is totally ordered.

<u>Proof of Lemma A.</u> The existence of $\mathcal{D}$ is by Zorn's

Lemma.

The set $P = \{ \mathcal{A} \mid \mathcal{A}$ is a collection of subsets of $X$ w/ f.i.p. $\}$.

The partial order on $P$ is containment.

$\mathcal{A} \subset \mathcal{B}$ if every subset of $X$ inside $\mathcal{A}$ is also in the collection $\mathcal{B}$.

Let us consider a chain $C$ in $P$

$\Rightarrow$ $C = \{ \mathcal{A}_1, \mathcal{A}_2, \ldots \mid \mathcal{A}_i$ is a collection of subsets of $X$ w/ fip $\}$

$\bigcup_{i \in I} \mathcal{A}_i$ is an upper bound for $C$.

$\Rightarrow$ By Zorn's Lemma, $P$ has a maximal element, say $\mathcal{D}$. ~~is~~ a collection of subsets of $X$ that satisfies f.i.p. $\boxtimes$

<u>Lemma B</u> Let $X$ be a set, $\mathcal{D}$ as before. Then

a) Any finite intersection of elements of $\mathcal{D}$ is also an element of $\mathcal{D}$.

b) If $A \subset X$ s.t. $A \cap D \neq \phi$ $\forall D \in \mathcal{B} \Rightarrow A \in \mathcal{B}$.

Proof of b). Let $A$ be as in b).

Define $\mathcal{E} = \mathcal{B} \cup \{A\}. \supset \mathcal{B}$.

We show that $\mathcal{E}$ also satisfies the f.i.p $\Rightarrow$ $\mathcal{E} = \mathcal{B} \Rightarrow A \in \mathcal{B}$.

by the maximality of $\mathcal{B}$,

Let $E_1, E_2, \dots, E_n \in \mathcal{E}$. If $E_i \neq A$, $i = 1, \dots, n$

$\Rightarrow E_i \in \mathcal{B} \Rightarrow \bigcap_{i=1}^{n} E_i \neq \phi$ as $\mathcal{B}$ satisfies

f.i.p.

If $E_1 = A$

then $A \cap E_2 \cap E_3 \dots \cap E_n \neq \phi$ as

$E_2 \cap E_3 \cap \dots \cap E_n \in \mathcal{B}$ (part (a))

$\Rightarrow \mathcal{E}$ satisfies f.i.p $\Rightarrow \mathcal{E} = \mathcal{B}$.

$\boxtimes$

## Proof of Tychonoff's Theorem

$X = \prod_{\alpha \in I} X_\alpha$ is compact in the product top.

Let $\mathcal{A}$ collection of subsets of $X$ having the f.i.p.

We'll prove that $\bigcap_{A \in \mathcal{A}} \bar{A} \neq \phi \Rightarrow X$ is compact.

By lemma A $\exists$ $\mathcal{B}$ of subsets of $X$ satisfying f.i.p., $\mathcal{B} \supset \mathcal{A}$. We'll prove

$$\bigcap_{D\in\mathcal{B}} \overline{D} \neq \phi. \qquad \text{———} \quad (1)$$

$\pi_\alpha : X \longrightarrow X_\alpha$ projection map. is continuous.

$\Rightarrow$ $\{ \pi_\alpha(D) \mid D \in \mathcal{B} \}$ of subsets of $X_\alpha$ has f.i.p. as $\mathcal{B}$ has f.i.p.

$\because$ $X_\alpha$ is compact we know that $\bigcap_{D\in\mathcal{B}} \overline{\pi_\alpha(D)} \neq \phi$.

$\Rightarrow$ $\forall \alpha \in I$, $\exists$ $x_\alpha \in \bigcap_{D\in\mathcal{B}} \overline{\pi_\alpha(D)}$.

Let $x = (x_\alpha)_{\alpha\in I} \in X$.

We'll show that $x \in \overline{D}$ $\forall D \in \mathcal{B}$

$\Rightarrow$ $x \in \bigcap_{D\in\mathcal{B}} \overline{D}$ $\Rightarrow$ prove (1).

if $\pi_\beta^{-1}(U_\beta)$ is any subbasis element containing $x$ then $\pi_\beta^{-1}(U_\beta)$ intersects every element of $\mathcal{B}$: $U_\beta$ is a nbd of $x_\beta$ in $X_\beta$

$\because$ $x_\beta \in \overline{\pi_\beta(D)}$ $\Rightarrow$ $U_\beta \cap \pi_\beta(D)$

$\Rightarrow \quad \pi_\beta^{-1}(U_\beta) \cap D \neq \phi \quad \forall \; D \in \mathcal{D}$.

$\Rightarrow$ by part b) of Lemma B we know that every subbasis element containing $x$ must lie in $\mathcal{D}$.

$\Rightarrow$ by part a) of Lemma B. every basis element of $X$ which contains $x$ also belong to $\mathcal{D}$.

$\therefore$ $\mathcal{D}$ has f.i.p $\Rightarrow$ every basis element in $X$ which contains $x$ intersects every element of $\mathcal{D}$. $\Rightarrow x \in \overline{D}$, $\forall \; D \in \mathcal{D}$ $\Rightarrow$ proves 1

$\Rightarrow$ the theorem. ∎