Problem Session 3

Problem Set 3

① compact and sequentially compact subsets of $\mathbb{R}_{cof}$.

All subsets are compact.

$A \subset \mathbb{R}$. Suppose $\{U_\alpha\}_{\alpha \in I}$ be an open cover of A $\Rightarrow$ U_α is open $\Rightarrow$ U_α^c is a finite set.

$\Rightarrow$ $A \setminus U_\alpha$ has only finitely many elements $x_1, x_2, \dots, x_n$. Choose $U_{x_i} \in \{U_\alpha\}_{\alpha \in I}$ s.t. $x_i \in U_{x_i}$. Then $U_\alpha \cup \{U_{x_i}\}_{i=1}^n$ is a finite sub cover of A. $\Rightarrow$ A is compact.

All subsets are sequentially compact.

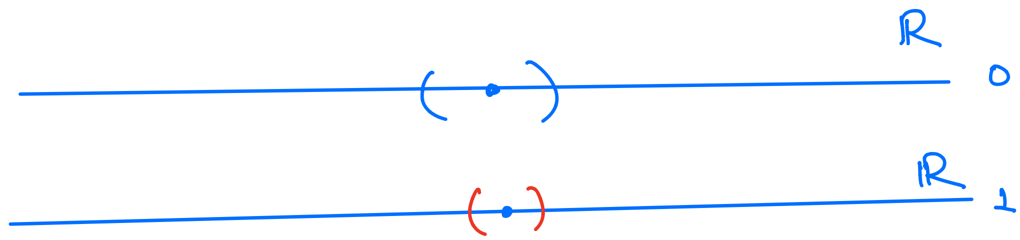
② Every compact subspace of a Hausdorff is closed.

X is non-Hausdorff but a subspace of X is Hausdorff.

$X = \{a, b, c\}$ $\quad \tau = \{X, \Phi, \{a\}, \{b\}, \{a, b\}\}$

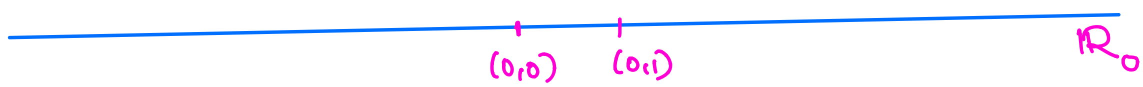
$A = \{a, b\}$

Line w/ two origins — Non-Hausdorff space.



$X = \mathbb{R} \times \{0,1\}$ $\quad (x,0)$ or $(x,1)$

$(x,0) = (x,1)$ unless $x = 0$. $\quad \tilde{X} = X/\sim$



$\tilde{X}$ is the line w/ two origins and is non-Hausdorff. as any open intervals containing $(0,0) \neq (0,1)$ must intersect non-trivially as all other points are identified.

(3) $\quad A \subset X \quad r : X \longrightarrow A$ continuous

s.t. $\quad r(a) = a \quad \forall a \in A$.

r is a quotient map.

U is open in $A \iff r^{-1}(U)$ is open in X.

Suppose U is open in A $\Rightarrow$ $r^{-1}(U)$ is open in X as r is continuous.

Assume for some $U \subset A$, $r^{-1}(U)$ is open in X.

We must prove $U = A \cap r^{-1}(U)$ is open in A.

$\Rightarrow$ r is a quotient map. $\boxtimes$

(4) X must be T_1 but not T_2.

$\mathbb{R}_{cof}$ is a T_1 space and is non-Hausdorff.

$\forall$ $x, y \in X$ $\exists$ $\underset{\text{open}}{U_x \ni x}$ and $\underset{\text{open}}{U_y \ni y}$ s.t.

$x \notin U_y$ and $y \notin U_x$.

Let $x, y \in \mathbb{R}$. Let U open containing x.

U^c is finite. So choose $U_x = \mathbb{R} \setminus \{y\}$.

$x \in U_x$ and $\mathbb{R} \setminus \{x\} = U_y$ is open s.t. $y \in U_y$

Clearly $x \notin U_y$ and $y \notin U_x$.

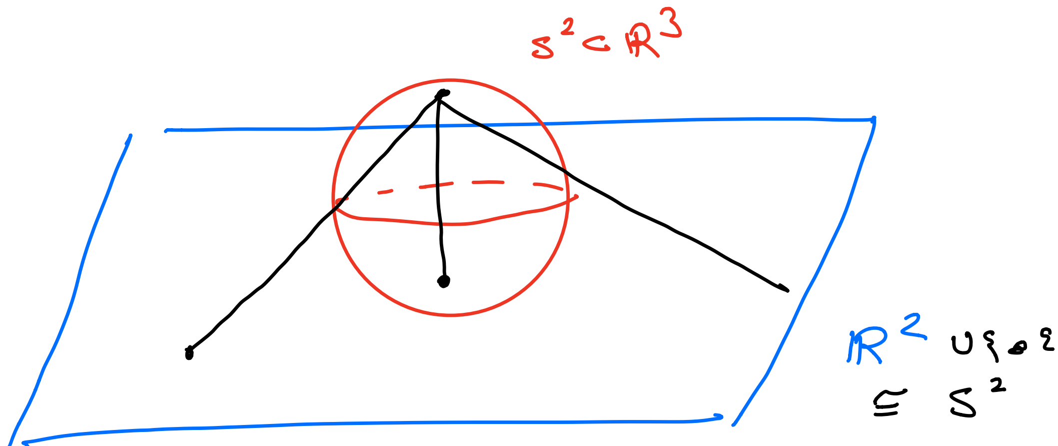
$\mathbb{R}_{cof}$ is not T_2.

$\underset{U_x}{\overset{x}{\cap}} \cap \underset{U_y}{\overset{y}{\cap}} \neq \phi$ as

$\mathbb{R}$ is uncountable and $U_x^{\,c}$ and $U_y^{\,c}$ are finite. $\boxtimes$

⑤. $X/A = X/\sim$

$D^n/S^{n-1} \cong S^n$.



stereographic projection.

$$f : \quad g = \frac{x}{1 - \|x\|} \qquad f = s \circ g$$

$$\begin{tikzcd}
D^n/S^{n-1} \arrow[r, "g"] \arrow[d, "\tilde{f}"'] & \mathbb{R}^n \arrow[dl, "s"] \\
S^n &
\end{tikzcd}$$

s : stereographic projection

define $f(S^{n-1}) = n \to$ north pole of S^n.

$$\tilde{f} : D^n/S^{n-1} \longrightarrow S^n$$
$$[x] \longrightarrow f(x)$$

from the theorem discussed in class, $\tilde{f}$ is a homeomorphism. ⊡

⑥ a) X is Hausdorff $\Leftrightarrow$
$\Delta = \{(x,x) \mid x \in X\} \subset X \times X$ is closed.

1b) i) $K = \{1/n \mid n \in \mathbb{N}\}$

$Y = \mathbb{R}_K \setminus K$ $\quad p: X \to Y$ quotient map.

Y is T_1 but not Hausdorff.

ii) Product of quotient maps need not be a quotient map.

$p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is NOT a quotient map.

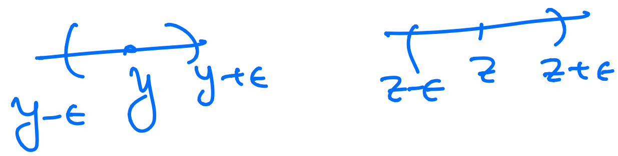
Open sets in $\mathbb{R}_K$ topology are unions of $(a,b) \cup (a,b) \setminus K$.

In Y $\quad [0] \neq [K]$ in Y

But $\exists$ no open sets in Y of $[0]$ and $[K]$ which do not intersect. NOT Hausdorff.

Any point in Y other than $[0]$, $[K]$ is $[y]$, $y \in \mathbb{R}$. $\quad [y] \quad [z]$

consider



y is T_1.