# Problem Session 2

1) $A \subset X$

i) $\mathring{A}$ is the largest open set contained in $A$.

Let $\exists$ $B$ s.t. $\underset{\text{open}}{B \subset A}$ and $\mathring{A} \subset B$.

$\exists$ $x \in B$ s.t. $x \notin \mathring{A}$.

$\because$ $B$ is open $\Rightarrow$ $\exists$ $U \ni x$ s.t. $U \subset B \subset A$

$\Rightarrow$ $x \in \mathring{A}$.

$\Rightarrow$ $\mathring{A} = \bigcup\limits_{\substack{U \subset A \\ \text{open}}} U$

ii) $\bar{A}$ is the smallest closed set that contains $A$.

Suppose $B$ is another closed set s.t.

$A \subset B$. $\Rightarrow$ $B \subset \bar{A}$ $\Rightarrow$ $\bar{A}^c \subset B^c$

$B^c$ is open and $\because$ $A \subset B \Rightarrow B^c \subset A^c$

Using the defn of closure, i.e., if $x \in \bar{A}$

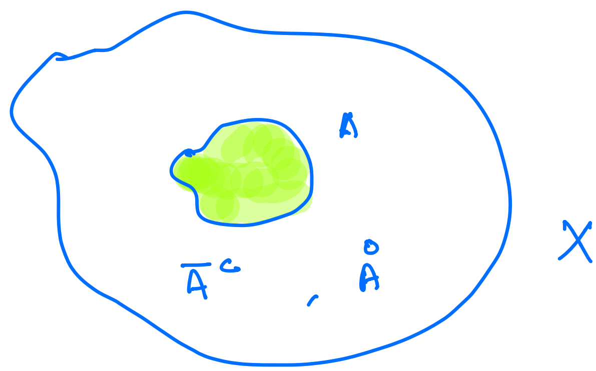
$\Rightarrow$ every open set containing $x$ must intersect

$A$ nontrivially.

$\Rightarrow$ every open set containing $x$ must intersect $B$ nontrivially $\Rightarrow$ we get that

$\overline{A} = B$.

$$\Rightarrow \quad \overline{A} = \bigcap_{\substack{C \supset A \\ \text{closed}}} C \quad \Rightarrow \quad \overline{A} \text{ is the smallest closed set containing } A.$$




$\partial A \cup \mathring{A} = \overline{A}$ ?

$\partial A = \{ x \in X \mid$ s.t. every nbd $U$ of $x$ must intersect $A$ and $A^c$ nontrivially$\}$

② $\mathbb{R}$

$\tau_1$ = standard top.

$\tau_2 = \mathbb{R}_\ell$

$\tau_3 = \mathbb{R}_u$ $\quad \tau_4$ = cofinite topology.

**Lemma** Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on $X$. Then TFAE.

1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

2) $\forall x \in X$ and $\forall B \in \mathcal{B}$ s.t $x \in B$, $\exists$ a basis element $B' \in \mathcal{B}'$ s.t $x \in B' \subset B$.

$\mathcal{T}_1 \qquad \subset \qquad \mathcal{T}_2$

$\mathbb{R}$, std $\qquad$ $\mathbb{R}$, lower limit top.

$\{[a, b) \mid b > a\}$

Suppose $x \in \mathbb{R}$ and $(a, b) \ni x$

$x \in [x, b) \in \mathcal{B}_l$ and $[x, b) \subset (a, b)$

$\Rightarrow$ from the lemma $\mathcal{T}_1 \subset \mathcal{T}_2$.

Can we say that $\mathcal{T}_1 \supset \mathcal{T}_2$?

Let $x \in \mathbb{R} \Rightarrow [x, b) \ni x$ and belongs to $\mathcal{B}_l$. elements in $\mathcal{B}_{std}$ are $(a, b)$.

$\Rightarrow$ from the lemma above

$\mathcal{T}_2 \supset \mathcal{T}_1$

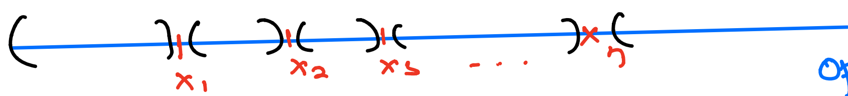
strict.

Similarly, $\tau_3 \underset{\text{strictly}}{\supset} \tau_1$.

$\tau_1 \underset{\text{strict.}}{\supset} \tau_4$

$(a,b)$ is NOT open in $\tau_4$ b/c

$(a,b)^c$ is not finite.




open set in $\tau_4$

$\tau_2$ incomparable $\tau_3$

$\mathbb{R}_l$ $\mathbb{R}_u$

$[a,b)$ $\nexists$ any element $\mathcal{B}_u$ which contains $a$ and is strictly contained in $[a,b)$.

$(x,a]$

Similarly $(a,b]$ in $\tau_3$

$\Rightarrow$ $\tau_2$ and $\tau_3$ are incomparable.

$T_2$ $\supset$ $T_4$

~~strict~~

$\mathbb{R}_\ell$ $\mathbb{R}_{\text{cofinite}}$.

$T_3$ $\supset$ $T_4$

~~strict~~.

(3)

(a) $\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{Q}\}$

generates the standard topology on $\mathbb{R}$.

$x \in \mathbb{R}$ and $(c, d) \ni x$

$\because$ $\mathbb{Q}$ is dense in $\mathbb{R}$

$\forall \epsilon > 0 \ \exists q \in \mathbb{Q}$ s.t. $|x - q| < \epsilon$.

$\exists \ q_1, q_2 \in \mathbb{Q}$ s.t. $x \in (q_1, q_2)$

and $(q_1, q_2) \subset (c, d)$

$\Rightarrow$ $\mathcal{B} = \mathcal{B}_{\text{standard}}$.

$\Rightarrow$ $\mathbb{R}$ has a countable basis.

(b). $\mathcal{C} = \{[a, b) \mid a, b \in \mathbb{Q}\}$

$\mathbb{R}_{\mathcal{C}} \underset{\text{strict}}{\subset} \mathbb{R}_\ell$

$\{[a,b) \mid a,b \in \mathbb{Q}\} \subset \{[c,d) \mid c,d \in \mathbb{R}\}$

$x \in \mathbb{R} \setminus \mathbb{Q}$ Then $[x,b) \in \mathcal{B}_\ell$ containing

$\mathbb{R} - \mathbb{Q}$

$x$ But no basis element from $\mathcal{C}$ can contain $x$ and be strictly contained inside $[x,b)$.

④ $\prod_{\alpha \in I} X_\alpha$ w/ product topology.

(a) $\{x_\alpha^n\}_{\alpha \in I} \longrightarrow \{x_\alpha\}_{\alpha \in I}$ in $\prod_{\alpha \in I} X_\alpha$

$\iff \quad x_\alpha^n \longrightarrow x_\alpha \quad \forall \alpha \in I.$

Why this won't hold for box topology.

$\mathbb{R}^\omega = \underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots}_{\text{countable product.}}$

w/ box topology.

Consider the sequence

$$x_\omega^n = \{ (1/n, 1/n, 1/n, \dots) \mid n \in \mathbb{N} \}$$

in $\mathbb{R}^\omega$.

$$\downarrow \text{ in } \mathbb{R}^\omega$$

$$(0, 0, 0, 0, \dots)$$

Consider the nbd

$$(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times (-1/4, 1/4) \times \dots (-1/n, 1/n) \times \dots$$

is a nbd of $(0, 0, \dots 0, \dots)$ in $\mathbb{R}^\omega$ w/ box topology. and $(x_\omega^n) \not\to (0, 0, \dots 0, \dots)$

4(b) similar approach as 4(a).

5(a) Box topology is finer than product.

6 $f : \mathbb{R}_{cof.} \to \mathbb{R}_{std}$ is continuous
$\Rightarrow$ $f$ is constant.

Suppose $f$ is NOT constant and $f(x) \neq f(y)$

for some $x \neq y$.

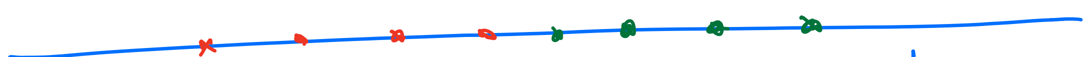
$\mathbb{R}_{std}$ is Hausdorff $\Rightarrow \exists\ U_x \ni f(x)$ and $U_y \ni f(y)$ $\underset{open}{\subset} \mathbb{R}$ w/ $U_x \cap U_y = \phi$.

$\because f$ is continuous $\Rightarrow \phi \neq f^{-1}(U_x)$ is open in $\mathbb{R}_{cof}$

$\phi \neq f^{-1}(U_y)$ is open in $\mathbb{R}_{cof}$.

$$f^{-1}(U_x) \cap f^{-1}(U_y) \neq \phi$$

$A, B \underset{open}{\subset} \mathbb{R}_{cof}. \Rightarrow A \cap B \neq \phi.$



a contradiction as $f^{-1}(U_x \cap U_y) = \phi$

$\Rightarrow f$ is constant. $\square$

5(b) $\{x_\alpha\} \in \prod X_\alpha$ can also be written as $f : I \to \prod X_\alpha$.

**Lemma** A $f_n \in X^I$ $\{f_n : I \to \prod X_\alpha \mid n \in \mathbb{N}\}$ converges in the box topology to $f \in X^I$ $\Leftrightarrow$ $\exists$ finite subset $J \subset I$ and an index $n_0 \in \mathbb{N}$ s.t.

$f_n|_J : J \to X \xrightarrow[\text{pointwise}]{}$ to $f|_J : J \to X$

and if $n > n_0$ and $\alpha \in I \setminus J$ $(I - J)$

then $f_n(\alpha)$ lies in every nbd of $f(\alpha)$.