Problem Session 1

PSet 1 :-

1. NOT every metric space comes from a norm.

<u>ex.</u> discrete metric space is NOT a normed space.

<u>Normed space</u> $\alpha \in \mathbb{R}$, $x \in X$

$$\|\alpha x\| = |\alpha| \, \|x\|$$

$\downarrow$

can never be bounded.

But there are bounded metric spaces, e.g., a discrete metric space.

2. every subset is open in (X, discrete metric).

$\underbrace{\{x\} \text{ is open in } X}_{}$ $\Rightarrow$ any subset is union of its elements $\Rightarrow$ open.

$$B_{1/2}(x) = \{x\}$$

any subset of a discrete metric space is closed.

every subset of a discrete metric space is both open and closed. $\rightsquigarrow$ connectedness.

any $x_n \to x$ is eventually constant.

$\epsilon-\delta$ def^n $\quad B_{1/2}(x)$

$\{x\}$ open and it contains x $\Rightarrow$ if $x_n \to x$ so $\exists n_0$ s.t $\forall n \geq n_0$, $x_n \in \{x\}$ $\Rightarrow$ $x_n = x$ $\forall n \geq n_0$

$\Rightarrow$ (x_n) is eventually constant.

(a) $f : (X, d) \to Y$ must be continuous.

$f^{-1}(U)$ is a subset of X $\Rightarrow$ open in X $\Rightarrow$ f is a continuous map.

open set $U \subset Y$ $\Rightarrow$

(b) $f : (\mathbb{R}^n, d_E) \to (X, d)$ is continuous

① then f must be constant.

if f is continuous and f is not constant

$f(x) \in X$ $\qquad X = \underbrace{\{f(x)\}}_{\substack{\text{open in } X \\ U}} \cup \underbrace{\{y \mid f(y) \neq f(x)\}}_{\substack{\text{open in } X \\ V}}$

$\mathbb{R}^n = f^{-1}(U) \cup f^{-1}(V)$ open

$f^{-1}(U) \cap f^{-1}(V) = \phi$

cannot happen b/c $\mathbb{R}^n$ is connected.

non-empty.

② any subset of X is both open and closed

$\Rightarrow$ if f is NOT a constant function then

for $A \subset X$

$f^{-1}(A)$ is open in $\mathbb{R}^n$ and $f^{-1}(A)$ closed in $\mathbb{R}^n$.

The only nonempty open and closed subsets are $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. $\Rightarrow$ $f^{-1}(A) = \mathbb{R}^n$

$\Rightarrow$ $f^{-1}(\{x\}) = \mathbb{R}^n$ $\Rightarrow$ $f(\mathbb{R}^n) = \{x\}$

$\Rightarrow$ f is a constant function.

$g : \mathbb{R} \to (X, d)$ g is continuous suppose $g^{-1}(A) = B$

B is again both open and closed, nonempty.

Suppose $B \neq \mathbb{R}$ $\Rightarrow$ $\exists$ some $y \notin B$.

$b_0 \in B$ assume $y > b_0$.

$Z = \{ x \in \mathbb{R} \mid x > b_0,\ x \notin B \}$

is bounded below by b_0, nonempty as $y \in Z$

$\Rightarrow$ by lub property $\exists$ $\inf Z = z$.

i) Suppose $z \in B$. $\because$ B is open $\Rightarrow$ $(z-\epsilon, z+\epsilon) \subset B$.

contradiction to the fact that $z = \inf Z$.

$z \notin B$.

ii) If $z \notin B$. $\because$ B is closed $\Rightarrow$ B^c is open

$\Rightarrow$ $\exists$ an open interval $(z-\epsilon, z+\epsilon) \subset B^c = \mathbb{R} \setminus B$.

$z - \epsilon/2$ contradicts the def of z being inf of Z.

$\therefore$ $B = \mathbb{R}$ $\Rightarrow$ $g : \mathbb{R} \to X$ must be constant.

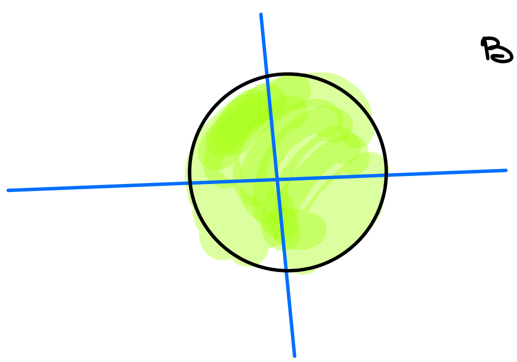
more open sets means fewer convergent sequences or fewer continuous functions.

③ Draw $B_1(0)$ in $(\mathbb{R}^2, d_1, d_2, d_\infty)$.

$B_1(0)$ in d_2 $= \{ (x,y) \in \mathbb{R}^2 \mid d_2((x,y),(0,0)) < 1 \}$

$= \{ (x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} < 1 \}$

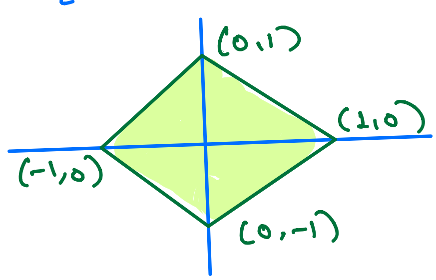
$= \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1 \}$



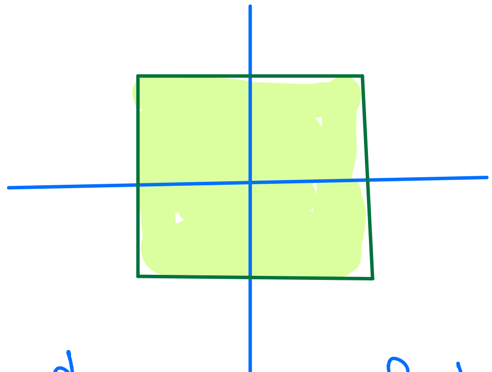
$B_1(0)$.

$B_1(0)$ in d_1 $= \{ (x,y) \in \mathbb{R}^2 \mid |x|+|y| < 1 \}$

$|x|+|y| = 1$



$B_1(0)$ in d_∞ = $\{(x,y) \in \mathbb{R}^2 \mid \max\{|x|,|y|\} < 1\}$



d_1, d_2 and d_∞ are equivalent metrics on $\mathbb{R}^2$.

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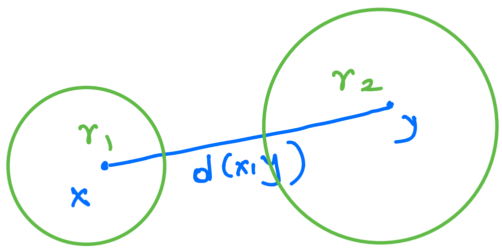


$r_1, r_2 < \frac{d(x,y)}{2}$

$x, y \in X$, dis $\Rightarrow$ $\{x\}$, $\{y\}$ are open sets.

5 $d'(x,y) = \min\{1, d(x,y)\}$ is a metric on X

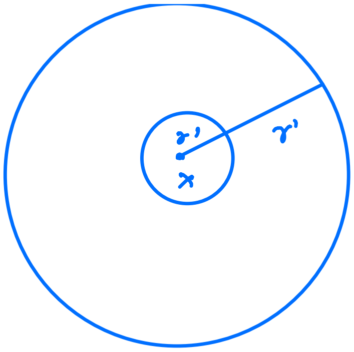
Let $x, y, z \in X$

$$\begin{aligned} d'(x,y) &= \min\{1, d(x,y)\} \\ &\leq \min\{1, d(x,z) + d(y,z)\} \\ &\leq \min\{1, d(x,z)\} + \min\{1, d(y,z)\} \\ &= d'(x,z) + d'(y,z) \end{aligned}$$

$\therefore$ d' is metric on X.

$B_r(x)$ in (X, d), $r < 1$

$B_r(x)$, $r < 1$ open in (X, d')



$\Rightarrow$ d and d' are equivalent

$\Rightarrow$ every metric space is equivalent to a bounded metric space

$\Rightarrow$ Boundedness is NOT a topological property.

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} < 1$$

, d' is also a metric on X.

⑥ (X, d_X) metric space.

$$X/\sim \; = \{ [x] \mid x \in X \}$$

<u>Remark</u>:- Quotient space of a metric space is NOT necessarily a metric space.

$$d([x],[y]) := \inf d_X(x,y) \quad , \; \begin{array}{l} x \in [x] \\ y \in [y] \end{array}$$

(a) d is a metric on $X/\sim$:-

If $[x], [y], [z] \in X/\sim$ $\quad \exists \; \begin{array}{l} x \in [x] \\ y \in [y] \\ z \in [z] \end{array}$

s.t.

$$d_X(x,y) = d([x],[y])$$
$$d_X(y,z) = d([y],[z])$$

i) $d([x],[y]) = d([y],[x])$ ✓

ii) $d([x],[y]) = 0 \iff [x]=[y]$?

$\Downarrow$

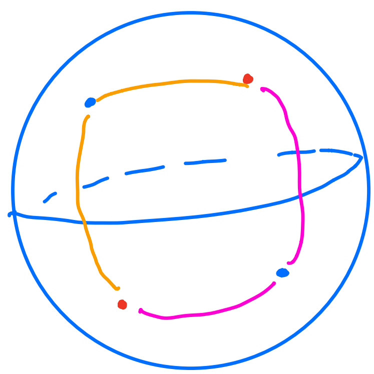
$d_X(x,y)=0 \iff x=y \iff [x]=[y]$.

iii) triangle inequality

$[x],[y],[z]$

$d([x],[y]) \le d([x],[z]) + d([y],[z])$

Prove this by contradiction.



Suppose $\exists$ $[x],[y],[z] \in X/\sim$

s.t. $d([x],[z]) > d([x],[y]) + d([y],[z])$

By the extra assumption $\exists$ $x' \in [x]$
$y' \in [y]$
and $[z'] \in [z]$

s.t. $d([x],[z]) = d_X(x',z')$ ...

$\Rightarrow$ $d([x],[z]) > d_X(x',y') + d_X(y',z')$

$\forall \; x'' \in [x], \; y'' \; [y], \; z'' \in [z]$

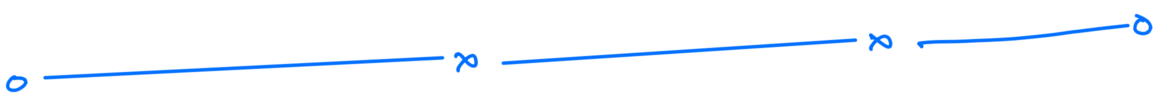
$$d_X(x'', z'') \leq d_X(x'', y'') + d_X(y'', z'')$$

$$\Rightarrow \inf \{ d_X(x'', z'') \mid z'' \in [z], \; x'' \in [x] \}$$

$$\leq \inf \{ \underline{\qquad\qquad\qquad} \}$$

$$\Rightarrow \inf_{\substack{x'' \in [x] \\ y'' \in [y] \\ z'' \in [z]}} \{ d_X(x'', y'') + d_X(y'', z'') \} > d_X(x', y') + d_X(y', z')$$

contradiction $\Rightarrow$ d satisfies the $\Delta \leq$.



$$f([a,b]) = \{x_1, \dots, x_n\}$$

$$\{x\}$$