Topology I
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Problem Set 7
Due date: 29.06.2021

Instructions

Problems marked with (∗) will be graded. Solutions may be written up in German or English (preferable) and should be handed in before the Problem sessions on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

(1) Let G be a group. The **commutator subgroup** $[G, G]$ of G is the subgroup generated by all elements of the form

$$[x, y] = xyx^{-1}y^{-1}$$

for $x, y \in G$.

(a) (∗) Show that $[G, G]$ is a normal subgroup of G and $[G, G] = e$ iff G is abelian.

(b) The **abelianization** of G is defined as the quotient group $G/[G, G]$. Show that this group is always abelian, and it is equal to G if G is already abelian.

(c) Given any two abelian groups G, H, find a natural isomorphism from the abelianization of the free product $G * H$ to the Cartesian product $G \times H$.

(d) (∗) Prove that the abelianization of $G = \{x, y \mid x^2 = y^3\}$ is isomorphic to $\mathbb{Z}$.

(2) (∗) Let Σ_g denote the surface of genus g. Prove that the abelianization of $\pi_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. Use this to prove that if g and h are non-negative integers with $g \neq h$ then Σ_g is not homeomorphic to Σ_h.

(3) Let σ be an n-simplex. Prove that σ is homeomorphic to the n unit ball B^n with Bd σ being carried to S^{n-1}.