Topology I — Humboldt-Universität zu Berlin
Shubham Dwivedi — Summer Semester 2021

Problem Set 5
Due date: 08.06.2021

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English (preferable) and should be handed in before the Problem sessions on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

(1) Show that if $h : S^1 \to S^1$ is nullhomotopic, then h has a fixed point and h maps some point $x \in S^1$ to its antipode $-x$.

(2) $(*)$ A map $h : S^n \to S^m$ is called **antipode-preserving** if $h(-x) = -h(x)$ for all $x \in S^n$.
We have the following fact: If $h : S^1 \to S^1$ is continuous and antipode-prserving, then h is not nullhomotopic. Use this fact to first prove:

There is no continuous antipode-preserving map $g : S^2 \to S^1$. Then prove that given a continuous map $f : S^2 \to \mathbb{R}^2$, there is point $x \in S^2$ such that $f(x) = f(-x)$. This is the **Borsuk-Ulam theorem** for S^2.

(3) For each of the following spaces, the fundamental group is either trivial, infinite cyclic or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternative holds: (Draw pictures!)
 (a) The "solid torus" $B^2 \times S^1$.
 (b) The torus T with a point removed.
 (c) The cylinder $S^1 \times \mathbb{R}$.
 (d) $\mathbb{R}^3$ with non-negative x, y and z axes removed.
 (e) $\{x \mid ||x|| > 1\} \subset \mathbb{R}^2$
 (f) $S^1 \cup (\mathbb{R}_+ \times 0) \subset \mathbb{R}^2$

(4) $(*)$ A space X is called **contractible** if the identity map of X to itself is nullhomotopic. Prove that X is contractible if and only if X has the homotopy type of a one-point space. Moreover, prove that a retract of a contractible space is contractible.

(5) Suppose $h : S^1 \to S^1$ is a continuous map. We define the **degree** of h as follows.
Let $b_0 = (1, 0) \in S^1$ and let $[\gamma]$ be the generator of $\pi_1(S^1, b_0)$. If $x_0 \in S^1$, choose a path α from b_0 to x_0 and define $\gamma(x_0) = \hat{\alpha}(\gamma)$. Then $\gamma(x_0)$ generates $\pi_1(S^1, x_0)$. Note that since $\pi_1(S^1)$ is abelian, $\gamma(x_0)$ is independent of the choice of path α.

Now for $h : S^1 \to S^1$, choose $x_0 \in S^1$ and let $h(x_0) = x_1$. Consider $h_* : \pi_1(S^1, x_0) \to \pi_1(S^1, x_1)$. Since both the groups are isomorphic to $(\mathbb{Z}, +)$, we have

$$h_*(\gamma(x_0)) = d \cdot \gamma(x_1) \tag{0.1}$$

for some integer d. The integer d is called the **degree of** h and is denoted by deg h.
deg h is independent of the choice of generator γ, choosing other generator merely changes the sign in (0.1).

 (a) Show that d is independent of the choice of x_0.
 (b) $(*)$ If $h, k : S^1 \to S^1$ are homotopic then deg $h =$ deg k, i.e., degree is homotopy invariant.
 (c) Show that deg $(h \circ k) =$ deg $(h) \cdot$deg (k).
 (d) $(*)$ Compute the degrees of the constant map, the identity map, $\rho(x_1, x_2) = (x_1, -x_2)$ and $h(z) = z^n$.