Topology I
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Problem Set 4
Due date: 01.06.2021

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English (preferable) and should be handed in before the Problem sessions on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

(1) Let x_0, x_1 be points of a path connected space X. Show that the isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is independent of path if and only if the fundamental group is abelian.

(2) $(*)$ Given two pointed spaces (X, p) and (Y, q), prove that $\pi_1(X \times Y, (p, q))$ is isomorphic to the product group $\pi_1(X, p) \times \pi_1(Y, q)$.

(3) For $A \subset X$, a **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a, \ \forall \ a \in A$. In this case, we say that A is a retract of X.
   (a) Prove that if $a_0 \in A$ then $r_* : \pi_1(X, a_0) \to (A, a_0)$ is surjective.
   (b) If A is a retract of X the homomorphism of fundamental groups induced by the inclusion $i : A \to X$ is injective.
   (c) $(*)$ Prove that there is no retraction of the unit ball B^2 onto S^1.

(4) Let $p : E \to B$ be a covering map with B connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$ then $p^{-1}(b)$ has k elements for all $b \in B$. In this case, E is called a $k-$fold cover of B.

(5) $(*)$ Let $p : E \to B$ be a covering map with E path-connected. Prove that if B is simply-connected then p is a homeomorphism.

(6) Prove that any covering map is a **local homeomorphism**, i.e., for a covering map $p : E \to B$, every point $e \in E$ has a neighbourhood that is mapped homeomorphically by p onto an open subset of B.
Prove that not every local homeomorphism is a covering map by showing that the map $p : \mathbb{R}_+ \to S^1$ given by
$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$
is a local homeomorphism but is not a covering map.