Topology I
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Problem Set 3
Due date: 011.05.2021

Instructions

Problems marked with ($*$) will be graded. Solutions may be written up in German or English (preferable) and should be handed in before the Problem sessions on the due date. For problems without ($*$), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

(1) What are all compact and sequentially compact subsets of $\mathbb{R}$ with cofinite topology?

(2) ($*$) Prove that every compact subspace of a Hausdorff space is closed. Can it happen that a space is non-Hausdorff but a subspace of it is Hausdorff?

(3) For $A \subset X$, a **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(a) = a,\ \forall\ a \in A$. Show that a retraction is a quotient map.

(4) ($*$) Give an example of a space with non-trivial topology which is T_1 but not T_2. (If you are giving Prob. 6 (b) (i) as an example then prove it.)

(5) The quotient of a space X by a subset $A \subset X$ is defined as $X/A := X/\sim$ with the quotient topology, where the equivalence relation is defined such that $x \sim y$ for every $x, y \in A$ and otherwise $x \sim x$ for all $x \in X$. In other words, X/A is the result of modifying X by "collapsing A to a point".

Show that for every $n \in \mathbb{N}$, S^n is homeomorphic to D^n/S^{n-1}, where $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

(6) (a) Prove that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$.

(b) Recall the space $\mathbb{R}_K$ with the K-topology from Problem set 2. Let $Y = \mathbb{R}_K/K$ and let $p : X \to Y$ be the quotient map. Prove that
(i) Y is T_1 but not Hausdorff.

(ii)($*$) $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map. (*Hint: Use part (a) to prove that* Δ *is not closed in* $Y \times Y$ *but its inverse image is closed in* $\mathbb{R}_K \times \mathbb{R}_K$.)