Topology I
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Problem Set 2
Due date: 04.05.2021

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English (preferable) and should be handed in before the Problem sessions on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

(1) Let X be a topological space and $A \subset X$. Prove the assertions made in the lecture that
  (a) The interior of A, $\mathring{A}$ is the largest open subset of X that is contained in A.
  (b) The closure of A, $\overline{A}$ is the smallest closed set of X that contains A.

(2) Consider $\mathbb{R}$. We define two new topologies on $\mathbb{R}$ by describing the basis as follows.

Consider the collection $\mathcal{B}$ of intervals of the form $[a, b),\ a < b$. The topology generated by $\mathcal{B}$ is called the **lower limit topology** on $\mathbb{R}$ and we denote the resulting topological space as $\mathbb{R}_l$.

Let $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and consider $\mathcal{B}'$ to be the collection of intervals of the form (a, b) along with all sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}'$ is called the $K-$**topology** on $\mathbb{R}$ and the resulting topological space is denoted as $\mathbb{R}_K$.

Compare the following topologies on $\mathbb{R}$, i.e., check which topology is weaker, stronger or incomparable :
  (a) $\mathcal{T}_1 =$ standard topology.
  (b) $\mathcal{T}_2 =$ lower limit topology.
  (c) $\mathcal{T}_3 =$ upper limit topology with basis consisting of intervals of the form $(a, b],\ a < b$.
  (d) $\mathcal{T}_4 =$ finite complement topology.

(3) (a) Show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a,\ b \in \mathbb{Q}\}$$

generates the standard topology on $\mathbb{R}$.
  (b) $(*)$ Show that the collection

$$\mathcal{C} = \{[a, b) \mid a,\ b \in \mathbb{Q}\}$$

as a basis generates a different topology than the lower limit topology on $\mathbb{R}$.

(4) Suppose $\{(X_a, \mathcal{T}_a)\}_{a \in I}$ is a (possibly infinite) collection of topological spaces and let $(X, \mathcal{T}) = \Pi_{a \in I} X_a$ with the product topology. Consider the subbase $\mathcal{B} \subset \mathcal{T}$ to consist of all sets of the form

$$U_b \times \Pi_{a \neq b} X_a \subset \Pi_a X_a$$

for arbitrary $b \in I$ and $U_b \in \mathcal{T}_{\lfloor}$. Prove:
  (a) $(*)$ A sequence $\{x_a^n\}_{a \in I}$ converges to $\{x_a\}_{a \in I}$ if and only if $x_a^n \to x_a$ for all $a \in I$.
  (b) For any other topological space Y, $f : Y \to X$ is continuous if and only if $\pi_a \circ f : Y \to X_a$ is continuous for every $a \in I$, where $\pi_a : X \to X_a$ denotes the projection $\{x_a\}_{a \in I} \mapsto x_a$.

(5) Assume I is an infinite set and $\{(X_a, \mathcal{T}_a)\}_{a \in I}$ is a collection of topological spaces. Consider the collection

$$\mathcal{B} = \{\Pi_{a \in I} U_a \mid\ U_a \in \mathcal{T}_a \text{ for all } a \in I\}.$$

the topology generated by $\mathcal{B}$ on $\Pi_{a \in I} X_a$ is called the **box topology**.
  (a) Compare the box topology and the product topology.

(b) What does it mean for a sequence in $\Pi_{a \in I} X_a$ to converge in the box topology?

(6) ($*$) Show that a map $f : \mathbb{R}_{\text{cof}} \to \mathbb{R}_{\text{std}}$ is continuous if and only if f is a constant function. Here $\mathbb{R}_{\text{cof}}$ denotes $\mathbb{R}$ with the cofinite topology and $\mathbb{R}_{\text{std}}$ with the standard topology.