Topology I
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Problem Set 1
Due date: 27.04.2021

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English (preferable) and should be handed in before the Problem sessions on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture.

Problems

(1) $(*)$ Prove that not every metric space comes from a norm, i.e., there are metric spaces whose metric are not induced by a norm.

Hint : Recall the properties of the norm and see if the metric from a norm could be bounded or unbounded?

(2) Show that on any set X with the discrete metric d, every subset is open.
Conclude that a sequence x_n converges to x if and only if $x_n = x$ for all n sufficiently large, i.e. the sequence is "eventually constant".
Then use this to prove the following statements:
(a) All maps from (X, d) to any other metric space are continuous.
(b) All continuous maps from $(\mathbb{R}^n, d_E)$ to (X, d) are constant. Here d_E is the Euclidean metric of $\mathbb{R}^n$ which we also denoted by d_2.

(3) $(*)$ Draw figures of the open unit ball $B_1(0)$ in the following spaces.
(a) $(\mathbb{R}^2, d_2)$
(b) $(\mathbb{R}^2, d_1)$
(c) $(\mathbb{R}^2, d_\infty)$.

(4) **Definition.** A *topological space* is called a **Hausdorff space** or is said to satisfy the Hausdorff property if for any two distinct points, there exist neighbourhoods of each which are disjoint from each other.
Prove that any metric space is a Hausdorff space. Give explicit disjoint open sets which you can use to separate points in a metric space with discrete metric.

(5) Show that for any metric space (X, d),

$$d'(x, y) = min\{1, d(x, y)\}$$

also defines a metric on X. Show that d and d' are equivalent. Conclude that every metric is equivalent to one that is bounded.

(6) Suppose (X, d_X) is a metric space and $\sim$ is an equivalence relation on X, with the resulting set of equivalence classes denoted by $X/\sim$. For equivalence classes $[x], [y] \in X/\sim$, define

$$d([x], [y]) := \inf d_X(x, y), \ x \in [x], \ y \in [y] \qquad (0.1)$$

(a)$(*)$ Show that d is a metric on $X/\sim$ if the following assumption is added: for every triple $[x], [y], [z] \in X/\sim$, there exist representatives $x \in [x], y \in [y]$ and $z \in [z]$ such that $d_X(x, y) = d([x], [y])$ and $d_X(y, z) = d([y], [z])$.
Comment: The hard part is proving the triangle inequality.

(b) Consider the real projective plane $\mathbb{RP}^2 = S^2/\sim$, where $S^2 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$ and the equivalence relation identifies antipodal points, i.e. $x \sim -x$. If d_X is the metric on S^2 induced by the standard Euclidean metric on $\mathbb{R}^3$, show that the extra assumption in part (a) is satisfied, so that (0.1) defines a metric on $\mathbb{RP}^2$.