Lecture 2

- Riemannian manifold
- Hopf fibration
- volume form

Local representation of metrics

Einstein Summation Convention

V^n - inner product space

we'll use subscripts for vectors in V: a basis of V is denoted by $e_1, \dots, e_n$

$\{e_i\}$

Let $v \in V$

$$v = \underline{\sum_{i=1}^{n} v^i e_i} = v^i e_i = [e_1, \dots, e_n] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

$$A^j e_j = A^1 e_1 + A^2 e_2 + \dots + A^n e_n.$$

Similarly, $\{e^i\}$ will denote the dual basis of

$V^* = \mathrm{Hom}(V, \mathbb{R})$ w/ $e^i(e_j) = \delta^i_j = \begin{cases} 1, & i=j \\ 0, & \text{otherwise} \end{cases}$

If $L: V \to V$ is a linear map then its matrix representation $[L^j_i]$

$$L(e_i) = L^j_i e_j$$

On manifolds, coordinate v.f. have subscripts

$$\partial_i = \frac{\partial}{\partial x^i} \qquad (x^i)$$

$\downarrow$

coordinate v.f.

dual covector field / 1-form dx^i and $dx^i(\partial_j) = \delta^i_j$

We can multiply 1-forms θ_1, θ_2 to get bilinear forms

$\theta_1 \cdot \theta_2 (u, v) = \theta_1(u) \cdot \theta_2(v) \neq \theta_2 \cdot \theta_1$

↳ tensor product of θ_1 and θ_2, $\theta_1 \cdot \theta_2 = \theta_1 \otimes \theta_2$.

$\therefore$ in local coordinates we get the bilinear forms

$dx^i dx^j = dx^i \otimes dx^j$

$g \to$ Riemannian metric

$g = g(\partial_i, \partial_j)\, dx^i dx^j$

as $\quad g(u, v) = g\left(dx^i(u)\partial_i, dx^j(v)\partial_j\right)$

$= g(\partial_i, \partial_j)\, dx^i(u) \cdot dx^j(v)$

$= g(\partial_i, \partial_j)\, dx^i dx^j$

$$\boxed{g = g_{ij}\, dx^i dx^j}$$

- expression of the metric in local coordinates using the Einstein Summation convention.

g_{ij} are the function which give a representation of g as a positive definite symmetric matrix.

$p \in M$, g_p i.p. on T_pM, $\{\partial_i\}$

$$g(\partial_i, \partial_j) = g(\partial_j, \partial_i)$$

$$\begin{bmatrix} & & \\ & (g_{ij}) & \\ & & \end{bmatrix}$$

e.g. $\mathbb{R}^n$, $g_{\text{Eucl.}}$ $\quad g_{ij} = \delta_{ij}$

$\therefore$ $\underline{g_{\text{Eucl.}}} = \delta_{ij}\, dx^i dx^j = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2$

$$= \sum_{i=1}^{n} (dx^i)^2.$$

<u>Exe.</u> $\mathbb{R}^2 \setminus \{\text{half line}\}$ $\quad (r, \theta)$ polar coordinates.

prove that the Euclidean metric on $\mathbb{R}^2 \setminus \{\text{half line}\}$ in polar coordinates is

$$g = dr^2 + r^2 d\theta^2$$

Tensors

$$TM\ ,\ T^*M$$

<u>Def</u>^{n}:- An (s,t)-tensor T is a section of the tensor bundle

$$\underbrace{TM \otimes TM \otimes \cdots \otimes TM}_{s\text{-times}} \otimes \underbrace{T^*M \otimes T^*M \otimes \cdots \otimes T^*M}_{t\text{-times}}$$

In local coordinates,

Select a frame $E_1, \ldots, E_n$ and construct the co-frame $\sigma^1, \ldots, \sigma^n$

$v = v^i E_i = \sigma^i(v) E_i$ – vectors

$\omega = \omega_j \sigma^j = \omega(E_j) e^j$

$$T = T^{i_1 \ldots i_s}_{j_1 \ldots j_t} E_{i_1} \otimes \cdots \otimes E_{i_s} \otimes \sigma^{j_1} \otimes \sigma^{j_2} \otimes \cdots \otimes \sigma^{j_t}$$

we'll simply write T in local coordinates as

$$T = T^{i_1 \cdots i_s}_{j_1, \cdots, j_t}.$$

$s+t$ is called the rank of the tensor.

a vector $E_i \rightsquigarrow$

$$w \longmapsto g(E_i, w)$$

$$g(E_i, w) = g(E_i, E_j)\sigma^j(w) = g_{ij}\sigma^j(w)$$

$\therefore$ the vector E_i can be converted to a covector σ^j using the Riemannian metric.

$$\underbrace{E_i}_{\text{vector}} \longmapsto \underbrace{g_{ij}\sigma^j}_{\text{covector}}$$

g^{ij} denotes the ij-th entry of the inverse matrix.

$$\sigma^j \longmapsto g^{ij} E_i.$$

$\sigma^i \mapsto E_i$, $\quad E_i \mapsto \sigma^i$

Remark :- when we use the coordinate v.f. as our frame then we need to invert g_{ij}.

However, if we use an orthonormal frame then $g^{ij} = g_{ij} =$ Identity matrix $\Rightarrow$ we can just move the indices up and down w/o any bother.

Examples :-

① Ricci tensor of g :- It's a $(1,1)$-tensor

$$Ric(E_i) = R_i{}^j E_j$$

$$\therefore \quad Ric = R_j{}^i \cdot E_i \otimes \sigma^j$$

$(1,1)$-tensor can be changed to a $(0,2)$-tensor

$$Ric = R_{jk} \cdot \sigma^j \otimes \sigma^k = g_{ji} R_k{}^i \cdot \sigma^j \otimes \sigma^k$$

also see this as a $(2,0)$-tensor

$$Ric = R^{ik} E_i \otimes E_k = g^{ij} R_j{}^k . E_i \otimes E_k .$$

② The curvature tensor g

(1,3)-tensor $R(X,Y)Z$

$$R = R_{ijk}^{\ell} \, E_\ell \otimes \sigma^i \otimes \sigma^j \otimes \sigma^k$$

can be converted into a (0,4)-tensor

$$R = R_{ijk\ell} \, \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \sigma^\ell$$

$$= R_{ijk}^{s} g_{s\ell} . \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \sigma^\ell$$

Remark :- Our convention is that

$$R_{ijk}^{s} g_{s\ell} = R_{ijk\ell}$$

There are some other conventions where the superscript comes to the 1st or 3rd place.

i.e, $R_{ijk}^{s} g_{s\ell} = R_{\ell ijk}$

$$R_{ijk}{}^{s} g_{sl} = R_{ijlk}$$

Contractions / Taking Trace of a tensor

$T = T_j{}^i \cdot E_i \otimes \sigma^j$ then the trace of T

$$\mathrm{Tr}\, T = C(T) = T_i{}^i$$

For a $(0,2)$-tensor T, the trace of T is

$$C(T) = T_{ik} g^{ik}$$

Trace of the Ricci curvature is called the scalar curvature

$$R = \mathrm{tr}(Ric) = g^{ij} R_{ij}$$
$$= R_{ijkl} g^{il} g^{jk}.$$

Inner product of tensors

$$|T|^2 = T_j{}^i T_l{}^k g^{lj} g_{ik}$$

Pointwise inner product

. T and S are (0,2)-tensors

$$\langle T, S \rangle = T_{ij} S_{kl} g^{ik} g^{jl}$$

for $S = T$

$$\langle T, T \rangle = |T|^2 = T_{ij} T_{kl} g^{ik} g^{jl}$$

L^2- inner product

$$(T, S) \qquad \int_M \langle T, S \rangle \, \mathrm{vol}.$$

↳ volume form.

→ Covariant Derivatives

→ Lie derivative

→ 1st order / 2nd order differential operators on (M, g)