Problem Set 4
Due date: 12.02.2024

Problems

(1) Let (M, g) be a compact, oriented manifold and $E \to M$ be a vector bundle with a fibre metric and a compatible connection. Prove that

$$\nabla^* \nabla s = -\mathrm{tr} \nabla^2 s \tag{0.1}$$

for $s \in \Gamma(E)$.

(2) Prove that for any $\omega \in \Omega^k(M)$

$$\Delta_d \omega = \nabla^* \nabla \omega + \mathrm{Ric}\omega. \tag{0.2}$$

where Ric denotes the Weitzenböck curvature operator.

(3) Let R be the Riemann curvature tensor of (M^n, g) and recall that it defines a symmetric bilinear map

$$R : \Lambda^2 TM \times \Lambda^2 TM \to \mathbb{R} \tag{0.3}$$

by

$$R\left(\sum X_i \wedge Y_i, \sum U_j \wedge V_j\right) = \sum R\left(X_i, Y_i, V_j, U_j\right) \tag{0.4}$$

and hence we can define the *curvature operator* $\mathfrak{R} : \Lambda^2 TM \to \Lambda^2 TM$ (which is self-adjoint) by

$$g\left(\mathfrak{R}\left(\sum X_i \wedge Y_i\right), \sum U_j \wedge V_j\right) = \sum R\left(X_i, Y_i, V_j, U_j\right). \tag{0.5}$$

(a) Prove that if $\mathfrak{R} \geq k$, $k \leq 0$, then $g\left(\mathrm{Ric}(T), T\right) \geq kC|T|^2\big)$, where C depends only on the type of the tensor.

(b) Prove that if $\mathfrak{R} \geq k$ and diam $(M) \leq D$ then for

$$V = \{T \in \Gamma(E) \mid \Delta_L T = \nabla^* \nabla T + c\mathrm{Ric}(T) = 0\} \tag{0.6}$$

we have

$$\dim V \leq m \cdot \exp\left(D \cdot C(n, kD^2) \frac{\sqrt{-kcC\nu}}{\sqrt{\nu} - 1}\right). \tag{0.7}$$

where ν is same as in Prob. 4, Pset 3 and m is the rank of E. Prove further, that if $k = 0$ and all $T \in V$ are parallel tensors.