Topics in Riemannian Geometry
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Problem Set 3
Due date: 29.01.2024

Problems

(1) Prove the statement from the lecture that if V is a finite-dimensional subspace of the space of sections of a vector bundle E of rank m over a compact manifold M and $p \geq 1$ is fixed then

$$\text{dim}V \leq m \cdot C_p(V)^p \tag{0.1}$$

where $C_p(V) = \sup\limits_{s \in V \setminus \{0\}} \dfrac{||s||_\infty}{||s||_{2p}}$.

(2) Let θ be a 1-form on (M, g) and X be its dual vector field. Prove that
  (a) $v \mapsto \nabla_v X$ is symmetric if and only if $d\theta = 0$.
  (b) $\text{div}X = -\delta\theta$.

(3) Let X be a vector field on (M, g) such that ∇X is symmetric. If $f = \frac{1}{2}|X|^2$ then prove:
  (a) $\nabla f = \nabla_X X$
  (b) $\Delta f = |\nabla X|^2 + \nabla_X(\text{div}X) + \text{Ric}(X, X)$.

(4) Prove the following result originally due to Gallot. Suppose (M, g) is compact such that

$$||u||_{2v} \leq S||\nabla u||_2 + ||u||_2$$

for some S and all smooth functions u and $v > 1$. For some vector bundle E over M, let $V \subset \Gamma(E)$ be finite dimensional. If

$$g(\nabla^*\nabla T, T) \leq \lambda|T|^2$$

for all $T \in V$ then

$$\text{dim}V \leq m \cdot \exp\left(\frac{S\sqrt{\lambda v}}{\sqrt{v} - 1}\right) \tag{0.2}$$