Topics in Riemannian Geometry
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Problem Set 2
Due date: 15.01.2024

Problems

(1) Prove that for vector fields X, Y, $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$.

(2) Show that a coordinate vector field ∂_k is a Killing field if and only if $\partial_k g_{ij} = 0$.

(3) Let X be a vector field on a Riemannian manifold.

   (a) Show that
$$|\mathcal{L}_X g|^2 = 2|\nabla X|^2 + 2\operatorname{tr}(\nabla X \circ \nabla X) \tag{0.1}$$

   (b) Prove the following integral formulas on a closed, oriented Riemannian manifold
$$\int_M \left(\operatorname{Ric}(X,X) + \operatorname{tr}(\nabla X \circ \nabla X) - (\operatorname{div} X)^2\right) = 0 \tag{0.2}$$
$$\int_M \left(\operatorname{Ric}(X,X) + g(\operatorname{tr}\nabla^2 X, X) + \frac{1}{2}|\mathcal{L}_X g|^2 - (\operatorname{div} X)^2\right) = 0. \tag{0.3}$$

   (c) Show that X is Killing if and only if
$$\operatorname{div} X = 0$$
$$\operatorname{tr}(\nabla^2 X) = -\operatorname{Ric}(X).$$

(4) A vector field X is said to be *affine* if $\mathcal{L}_X \nabla = 0$. Prove that Killing fields are affine. Also, prove the following result, originally due to Yano: If X is an affine vector field then show that $\operatorname{tr}(\nabla^2 X) = -\operatorname{Ric}(X)$ and that $\operatorname{div} X$ is constant. Use this together with the above characterizations of Killing fields to show that on closed manifolds affine fields are Killing fields.