Problem Set 1
Due date: 11.12.2023

Problems

(1) Find the expression of the $(1,3)$ Riemann curvature tensor in coordinates in terms of the Christoffel symbols. (**Hint:** Proceed in the same as we obtained a formula for the Christoffel symbols in terms of the derivatives of the components of the metric.)

(2) Prove the following
  (a) Ric is a symmetric $(0,2)$-tensor.
  (b) $\nabla_i R^i{}_{jmk} = \nabla_k R_{jm} - \nabla_m R_{jk}$.
  (c) $\nabla_i R^i_j = \frac{1}{2}\nabla_j \mathrm{R}$ which in invariant form reads as $\operatorname{div}\mathrm{Ric} = \frac{1}{2}d\,\mathrm{R}$ where div is the divergence of a tensor.

(3) Let g be a Riemannian metric on a manifold M. Let $c > 0$ be a constant. Then cg is again a Riemannian metric on M as is said to be a rescaled metric. Show that as a $(1,3)$-tensor $R_{(1,3)}(cg) = R_{(1,3)}(g)$. Find the scalings of the $(0,4)$-Riemann curvature tensor, the Ricci curvature and the Scalar curvature. (**Hint:** What can you say about the Levi-Civita connection ∇^{cg} in terms of ∇^g?)

(4) Prove the **Bochner formula** for $|\nabla f|^2$, i.e., for $f \in C^\infty(M)$, prove that

$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla^i f\nabla^j f + 2\nabla_i f\nabla^i(\Delta f). \tag{0.1}$$

Conclude from this that if $\mathrm{Ric} \geq 0$, $\Delta f = 0$ and $|\nabla f| = \text{constant}$ then ∇f is parallel.

(5) Prove the following integral equality:

$$\int_M |\nabla\nabla f|^2 \,\mathrm{vol} + \int_M \mathrm{Ric}(\nabla f, \nabla f)\,\mathrm{vol} = \int_M (\Delta f)^2\,\mathrm{vol} \tag{0.2}$$

and using the fact that[1], $|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2$, show that

$$\int_M \mathrm{Ric}(\nabla f, \nabla f)\,\mathrm{vol} \leq \frac{n-1}{n}\int_M (\Delta f)^2\,\mathrm{vol}\,. \tag{0.4}$$

(**Hint:** Integration by parts!)

(6) Use the above to prove the following theorem due to **Lichnerowicz.** Suppose f is an eigenfunction of Δ with eigenvalue $\lambda > 0$, i.e., $\Delta f + \lambda f = 0$. If $\mathrm{Ric} \geq (n-1)K$ for some constant $K > 0$ then $\lambda \geq nK$.

(7) The purpose of this problem is to show that in dimension 3, the Ricci curvature determines the Riemann curvature tensor.
  (a) Let (M^3, g) be a 3-dimensional Riemannian manifold and let us diagonalize the curvature operator (as a self-adjoint operator on 2-forms) Rm with respect to a basis $\{e_2\wedge e_3, e_3\wedge e_1, e_1\wedge e_2\}$ of $\Lambda^2 T^*M$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of TM^3 (this is possible because Rm is self-adjoint). Suppose that, with respect to this basis, Rm is a diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3$ down the diagonal. Then with respect to the basis $\{e_1, e_2, e_3\}$, prove that the Ricci tensor takes the form

$$\mathrm{Ric} = \frac{1}{2}\begin{bmatrix}\lambda_2+\lambda_3 & 0 & 0\\ 0 & \lambda_3+\lambda_1 & 0\\ 0 & 0 & \lambda_1+\lambda_2\end{bmatrix} \tag{0.5}$$

[1]This is the usual Cauchy-Schwarz inequality. More generally, if S is any $(2,0)-$tensor then

$$|S_{ij}|^2_g \geq \frac{1}{n}(g^{ij}S_{ij})^2 \tag{0.3}$$

and the scalar curvature $R = \lambda_1 + \lambda_2 + \lambda_3$. (**Hint:** Use the geometric interpretation of the Ricci and scalar curvatures.)

(b) Prove that an Einstein metric on a manifold of dimension $n \geq 3$ has constant scalar curvature. If $n = 3$, the metric has constant sectional curvature.