Lecture 3

Lie Derivatives

Suppose $f: M \to \mathbb{R}$ is a function and let $Y \in \Gamma(TM)$ then the directional derivative of f in the direction of Y is

$$D_Y f = \mathcal{L}_Y f = df(Y) = Y(f)$$

in local coordinates,

$$df = \partial_i f \, dx^i .$$

We do not need a metric g for defining df.

If we do have a metric g then we can define gradient of $f = \nabla f$ s.t.

$$g(V, \nabla f) = df(V) \quad \forall \; V \in \Gamma(TM).$$

In local coordinates, $\nabla f = g^{ij} \partial_i(f) \partial_j$.

We cannot define ∇f w/o g.

If $X \in \Gamma(TM)$ and F^t corresponding locall defined flow of X on M

$\Rightarrow$ for small t, $t \longmapsto F^t(p)$ is the integral curve of X through p at $t=0$.

<u>Def^n</u>:- The Lie derivative of a tensor in the direction of X is the 1^{st} order term in a suitable Taylor series expansion of the tensor when it is moved by the flow of X.

<u>for functions</u>

$$f(F^t(p)) = f(p) + t\,(\mathcal{L}_X f)(p) + \text{h.o.t.}$$

$$\Rightarrow \quad (\mathcal{L}_X f)(p) = \lim_{t \to 0} \frac{f(F^t(p)) - f(p)}{t}$$

$= df(X).$

We want to find out what $\mathcal{L}_X Y$ is.

$Y_{|F^t}$ can't be compared to Y b/c they live in different vector spaces.

$\therefore$ we look at $t \longmapsto DF^{-1}\left(Y_{|F^t(p)}\right) \in T_pM.$

So the Taylor series expansion in t at 0 gives

$$DF^{-1}\left(Y_{|F^t(p)}\right) = Y_{|p} + t(\mathcal{L}_X Y)\Big|_p + \text{hot}$$

and $\therefore$ we can define

$$(\mathcal{L}_X Y)\Big|_p = \lim_{t \to 0} \frac{DF^{-1}\left(Y_{|F^t(p)}\right) - Y_{|p}}{t}$$

<u>Prop</u> $\mathcal{L}_X Y = [X, Y].$

Now suppose T is a $(0,k)$-tensor. We define

$$(F^t)^* T = T + t(\mathcal{L}_X T) + hot$$

$\Longrightarrow$

$$((F^t)^* T)(Y_1, \dots, Y_k) = T(DF^t(Y_1), \dots, DF^t(Y_k))$$

$$= T(Y_1, \dots, Y_k) + t(\mathcal{L}_X T)(Y_1, \dots, Y_k) + hot$$

$$\therefore (\mathcal{L}_X T)(Y_1, \dots, Y_k) = \lim \frac{(F^t)^* T - T}{t}$$

Prop

$$(\mathcal{L}_X T)(Y_1, \dots, Y_k) = D_X(T(Y_1, \dots, Y_k)) - \sum_{i=1}^{k} T(Y_1, \dots, \mathcal{L}_X Y_i, \dots, Y_k).$$

Prop $\mathcal{L}_X(T_1 \otimes T_2) = (\mathcal{L}_X T_1) \otimes T_2 + T_1 \otimes \mathcal{L}_X T_2.$

Prop If a v.f. X vanishes at $p \in M$, then the value of $\mathcal{L}_X T$ at p depends only on the value at p.

Remark :- We didn't need g to define $\mathcal{L}_X(\cdot)$.

Lie Derivative and the metric

We can define the Hessian of a function on (M, g) using the Lie derivative.

Hess f is a $(0,2)$-tensor and is defined as

$$\boxed{\operatorname{Hess} f(X, Y) = \frac{1}{2}\left(\mathcal{L}_{\nabla f}\, g\right)(X, Y).}$$

We can also define $\operatorname{div} X$ using the Lie derivative

Defn The divergence of a v.f. X is a function $\operatorname{div} X$ on M^n that measures how the volume form of M changes along the flow for X.

$$\mathrm{vol} \in \Gamma(\Lambda^n T^* M)$$

$$\mathcal{L}_X \mathrm{vol} = (\mathrm{div} X) \mathrm{vol}$$

The Laplacian of a function Δf is the div. of the v.f. ∇f, i.e,

$$\Delta f = \mathrm{div}(\nabla f) = \mathrm{div}(df)$$

In fact, Δf = trace of the Hessian f

If $\{E_i\}$ is a positively oriented o.n frame, then

$$\mathrm{div} X = (\mathcal{L}_X \mathrm{vol})(E_1, \dots, E_n)$$

$$= \mathcal{L}_X (\mathrm{vol}(E_1, \dots, E_n))$$

$$- \sum_i \mathrm{vol}(E_1, \dots, \mathcal{L}_X E_i, \dots, E_n)$$

$$= 0 - \sum_{i=1}^{n} g(\mathcal{L}_X E_i, E_i)$$

$$= \frac{1}{2} \sum \left(\mathcal{L}_X (g(E_i, E_i)) - g(\mathcal{L}_X E_i, E_i) - g(E_i, \mathcal{L}_X E_i) \right)$$

$$= \sum \frac{1}{2} (\mathcal{L}_X g)(E_i, E_i)$$

$\therefore$

$$\operatorname{div} X = \frac{1}{2} \sum_{i=1}^{n} (\mathcal{L}_X g)(E_i, E_i)$$

when $X = \nabla f$

$$\Rightarrow \quad \Delta f = \operatorname{div}(\nabla f) = \frac{1}{2} \sum_{i=1}^{n} (\mathcal{L}_{\nabla f}\, g)(E_i, E_i)$$

$$= \operatorname{trace}(\operatorname{Hess} f).$$

The Covariant Derivative

Defⁿ Let $E \xrightarrow{\pi} M$ be a vector bundle. A <u>connection</u> on E is a map

$$\nabla : \Gamma(M) \times \Gamma(E) \longrightarrow \Gamma(E) \text{ s.t.}$$

1) $\nabla_X Y$ is $C^\infty(M)$-linear in X.

2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y.

3) For $f \in C^\infty(M)$, ∇ satisfies the Leibniz rule

$$\nabla_X (fY) = X(f)\, Y + f\, \nabla_X Y.$$

$\nabla_X Y$ is the covariant derivative of Y in the direction of X.

∇ on E is completely determined by its Christoffel symbols Γ_{ij}^{k} (in local coordinates)

$$\{\partial_i\} \quad \{E_j\}$$

$$\nabla_{\partial_i} E_j = \Gamma_{ij}^{k} E_k$$

Lemma :- If $E = TM$ is the tangent bundle then we can define connections on all tensor bundles $T^k_\ell(M)$ s.t.

1) $\nabla_X f = X(f) = df(X) = \mathcal{L}_X f$.

2) $\nabla_X (T \otimes S) = (\nabla_X T) \otimes S + T \otimes \nabla_X S$.

3) $\nabla_X (\mathrm{tr}\, T) = \mathrm{tr}(\nabla_X T)$, i.e, trace commutes w/ taking covariant derivative.

We'll see in the next lecture: →

- Levi-Civita connection on (M, g)
- Curvature of the Levi-Civita connection and its properties.