The covariant derivative

To differentiate tensors we need a connection.

Def^n:- Let $E \xrightarrow{\pi} M$ be a v.b. A connection on E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \text{ s.t.}$$

1) $\nabla_X Y$ is $C^\infty(M)$-linear in X.

2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y.

3) For $f \in C^\infty(M)$, ∇ satisfies the Leibniz rule

$$\nabla_X (fY) = X(f)Y + f\nabla_X Y.$$

$\nabla_X Y$ is the covariant derivative of Y in the direction of X.

∇ on E is completely determined by its Christoffel symbols Γ_{ij}^k which in local coordinates can be defined as

$$\nabla_{\partial_i} E_j = \Gamma_{ij}^k E_k.$$

Lemma:- If TM is the tangent bundle then we can define connections on all tensor bundles $T_l^k(M)$ s.t.

1. $\nabla_X f = X(f)$.
2. $\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$.
3. $\nabla_X (\operatorname{tr} Y) = \operatorname{tr}(\nabla_X Y)$. for all traces over any index of Y.

In local coordinates

$$(\nabla_X F) = (\nabla_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k}) \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k} X^p$$

and also

$$\nabla_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k} = \partial_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k} + \sum_{s=1}^{l} F^{j_1 \cdots q \cdots j_l}_{i_1 \cdots i_k} \Gamma^{j_s}_{pq} - \sum_{s=1}^{k} F^{j_1 \cdots j_l}_{i_1 \cdots q \cdots i_k} \Gamma^{q}_{p i_s}.$$

<u>Defn</u> <u>Gradient</u>

Let $f \in C^\infty(M)$. $df \in \Gamma(T^*M)$

$(df)^\# \in \Gamma(TM)$ is called the <u>gradient</u> of f w.r.t. g and is denoted by ∇f.

in local coordinates, $df = \frac{\partial f}{\partial x^j} dx^j$

$$(\nabla f) = (\nabla f)^i \frac{\partial}{\partial x^i}$$

$$= \left(g^{ij} \frac{\partial f}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

<u>example</u> S^2 w/ spherical coordinates.

round metric on S^2, $g = (d\phi)^2 + \sin^2\phi \, (d\theta)^2$ in these coordinates.

$$\nabla f = \frac{\partial f}{\partial \theta} g^{\theta\theta} \frac{\partial}{\partial \theta} + \frac{\partial f}{\partial \phi} g^{\phi\theta} \frac{\partial}{\partial \theta}$$

$$+ \frac{\partial f}{\partial \phi} g^{\theta\phi} \frac{\partial}{\partial \phi} + \frac{\partial f}{\partial \phi} g^{\phi\phi} \frac{\partial}{\partial \phi}$$

and $g\left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\right) = 1$, $g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right) = 0$

$$g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = \sin^2\phi$$

The Levi-Civita Connection

Let (M^n, g) Riemm. mfld.

__Defn__ A connection ∇ on TM is said to be compatible with g if

$$\nabla g = 0.$$

(ie g is parallel)

If $\nabla g = 0 \Rightarrow \nabla_X g = 0 \quad \forall X$

$\Leftrightarrow (\nabla_X g)(Y, Z) = 0 \quad \forall Y, Z,$

$$\Leftrightarrow X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

In local coordinates,

$$\left(\nabla_{\frac{\partial}{\partial x^k}} g\right)_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}$$

$\therefore \nabla g = 0 \Leftrightarrow$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} \quad \forall i, j, k$$

Recall :→ The torsion T^∇ of a connection ∇ on TM is

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Thm [Fundamental Theorem of Riemannian Geometry]

Let (M^n, g) be Riemm. Then $\exists!$ connection ∇ that is both metric compatible and torsion-free. ∇ is called the Levi-Civita connection.

Proof :→ We'll show that it must be unique if it exists, by deriving a formula for it (Koszul formula).

Let $X, Y, Z \in \Gamma(TM)$

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$y(g(x,z)) = g(\nabla_y z, x) + g(z, \nabla_y x)$$

$$z(g(y,x)) = g(\nabla_z y, x) + g(y, \nabla_z x)$$

and $\because$ $T^{\nabla} = 0$

$$\Rightarrow \quad \nabla_x y - \nabla_y x = [x,y]$$

$$\nabla_z x - \nabla_x z = [z,x]$$

$$\nabla_y z - \nabla_z y = [y,z]$$

$\therefore$ we get

$$x(g(y,z)) + y(g(x,z)) - z(g(x,y))$$

$$= 2g(\nabla_x y, z) + g(y,[x,z]) + g(z,[y,x]) - g(x,[z,y])$$

$$\Rightarrow g(\nabla_x y, z) = \frac{1}{2}\Big[x(g(y,z)) + y(g(x,z)) - z(g(x,y)) - g(y,[x,z]) - g(z,[y,x]) + g(x,[z,y]) \Big]$$

So $\nabla_X Y$ is determined uniquely.

Define ∇ by this formula and show that ∇ is compatible and torsion free.

- in local coordinates, the Christoffel symbols of ∇^{LC} are (for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$)

$$\Gamma_{ij}^{m} g_{mk} = \frac{1}{2}\left[\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij}\right]$$

$$\boxed{\Rightarrow \Gamma_{ij}^{k} = \frac{1}{2} g^{kl}\left[\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right]}$$

We'll use this formula frequently.

Orientation

If M is orientable, then a choice of such a cover or equivalently, a choice of nowhere-zero n-form) is called an orientation for M.

Such a form μ is called a volume form on M. Two volume forms $\mu, \tilde{\mu}$ corresponding to the same orientation $\iff$ $\mu = f\tilde{\mu}$ for some $f \in C^\infty(M)$ s.t. f is everywhere positive.

Let M be orientable and have k-connected components then $\exists\ 2^k$ orientations on M.

If M^n is oriented, compact, we can integrate n-forms on M. $\int_M \omega \in \mathbb{R}$

$\omega \in \Omega^n(M)$

Stokes' Theorem If $\partial M = \phi$

then $\int_M d\sigma = 0$

If $F : M \xrightarrow{\text{diffeo}} N$

$\omega \in \Omega^n(N) \Rightarrow F^*\omega \in \Omega^n(M)$

$$\Rightarrow \boxed{\int_M F^*\omega = \int_{N=F(M)} \omega}$$

Defn:- A manifold w/ volume form is an oriented mfld M together w/ a particular choice μ (representative of the equivalence-class of the orientation).

If M is compact the we can integrate functions on M by defining

$$\int_M f := \int_M f\mu$$

whose value depends on the choice of μ

Let (M,μ) be a manifold w/ volume form define the divergence $\mathrm{div} : \Gamma(TM) \longrightarrow C^\infty(M)$
Linear

by
$$\mathcal{L}_X \mu = d(X \lrcorner \mu) + \underbrace{X \lrcorner d\mu}_{=0}$$
$$= (\mathrm{div}\, X)\,\mu$$

(Depends on μ)

<u>Notice</u> :- $\operatorname{div} X = 0 \iff \mathcal{L}_X \mu = 0$

$\iff \theta_t^* \mu = \mu$ where

θ_t is the flow of X.

$\iff$ μ is invariant under flow of X.

If M compact,

$$\operatorname{vol}(M) = \int_M 1 = \int_M 1 \cdot \mu$$

Suppose $\operatorname{div} X = 0 \implies$

$$\int_{\theta_t(M)} \mu = \operatorname{vol}(\theta_t(M)) = \int_M \theta_t^* \mu = \int_M \mu = \operatorname{Vol}(M)$$

$\implies \operatorname{vol}(\theta_t(M)) = \operatorname{vol}(M)$

$\therefore$ flow of a Divergence-free v.f. preserves the volume.

Divergence Theorem

Let $X \in \Gamma(TM)$, (M, μ) be compact

then $\int_M (\operatorname{div} X) = 0$ as

$$\int_M (\operatorname{div} X)\mu = \int_M d(X \lrcorner \mu) = 0 \quad \text{by Stokes' Thm.}$$

Let (M, g) be an oriented Riemannian manifold. Then $\exists$ a canonical volume form μ on (M, g) defined by the requirement that

$$\mu(e_1, \dots, e_n) = 1 \quad \text{whenever } \{e_1, \dots, e_n\}$$

is an oriented orthonormal basis of (T_pM, g_p).

i.e., given a local oriented o.n. frame for M $\{e_1, \dots, e_n\}$,

$$\mu = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

$\mu = \sqrt{\det g}\; dx^1 \wedge \cdots \wedge dx^n$ in any local coordinates $(x^1, x^2, \dots, x^n)$.

- Divergence theorem holds for any manifold w/ volume $\Rightarrow$ also holds for oriented Riemm. vol. form and symplectic manifolds.

Curvature of the Levi-Civita connection

We call R, as the Riemann curvature tensor of

g.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= -R(Y,X)Z$$

<u>Remark</u> :- $R^{\nabla} = 0$ and $T^{\nabla} = 0$ iff $\exists$ <u>local parallel coordinate frames.</u>

One defn of being <u>flat</u> for any connection is $R^{\nabla} = 0$

and for a Riem. mfld we defined <u>flat</u> as "locally isometric" to $(\mathbb{R}^n, \hat{g})$.

For the Riemannian curvature of Levi-Civita conn, the two notions of <u>flatness are the same.</u>

Symmetries of R

$$R(X,Y,Z,W) := g(R(X,Y)Z, W)$$

$\downarrow$

(4,0) tensor obtained from (3,1) R by musical isomorphisms.

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}^{\ \ \ \ l}\partial_l$$

$$R(\partial_i, \partial_j, \partial_k, \partial_m) = R_{ijkm}$$

$$\boxed{R_{ijkm} = R_{ijk}^{\ \ \ \ l} g_{lm}}$$

Prop :-

a) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

b) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

c) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W)$

$= 0$

d) $R(X,Y,Z,W) = R(Z,W,X,Y)$.

a) is always true and w/ b) allows us to see $R \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$ i.e, as a symmetric bilinear forms on the space of 2-forms.

b) follows from metric compatibility, $\nabla g = 0$

c) is true for any torsion free connection on TM. It is called the First Bianchi identity.

d) follows from a), b) and c).

Proof :- a) done

b) since $\nabla g = 0 \Rightarrow$

$$Y(g(Z,Z)) = 2g(\nabla_Y Z, Z)$$

$$X(Y(g(Z,Z))) = 2X(g(\nabla_Y Z, Z))$$

$$= 2g(\nabla_X \nabla_Y Z, Z) \quad \text{--- (1)}$$

$$+ 2g(\nabla_y Z, \nabla_x Z)$$

$$Y(X(g(Z,Z))) = 2g(\nabla_y Z, \nabla_x Z) + 2g(Z, \nabla_y \nabla_x Z) - ②$$

$$[X,Y](g(Z,Z)) = 2g(Z, \nabla_{[X,Y]} Z) - ③$$

① + ② − ③

$$X(Y(g(Z,Z))) - Y(X(g(Z,Z))) - [X,Y](g(Z,Z)) = 0$$

$$= 2R(X,Y,Z,Z) = 0$$

$\Rightarrow$ polarize to get (b).

c) Want to show that

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

expand and use torsion-free. and use

Jacobi identity for $[\cdot,\cdot]$.

d) Write identity c) in 4 ways.

Sectional Curvature

Let (M,g) be Riemann.

Given $X_p, Y_p \in T_pM$

$$|X_p \wedge Y_p|^2_{g_p} = |X_p|^2_{g_p} |Y_p|^2_{g_p} - g_p(X_p, Y_p)^2$$

$$|X \wedge Y|^2 = |X|^2 |Y|^2 - \langle X, Y\rangle^2$$

Defn:- Let L_p be a 2-dimensional subspace of T_pM $(n \geq 2)$. Define the sectional curvature $K_p(L_p)$ of (M,g) at p in

"L_p direction" by

$$K_p(L_p) = \frac{R(X_p, Y_p, Y_p, X_p)}{|X_p \wedge Y_p|^2}$$

for any basis X_p, Y_p of L_p.

(denom. not zero as X_p, Y_p are basis).

if $\tilde{X} = aX + bY$

$\tilde{Y} = cX + dY$

$\tilde{X} \wedge \tilde{Y} = (ad - bc)\, X \wedge Y$

Show that

$$\frac{R(\tilde{X}, \tilde{Y}, \tilde{Y}, \tilde{X})}{|\tilde{X} \wedge \tilde{Y}|^2} = \frac{R(X, Y, Y, X)}{|X \wedge Y|^2}$$

If $n = 2$, $L_p = T_pM \quad \forall p \in M$

$\Rightarrow$ sectional curvature is just a smooth function on M.

Lemma :- The sectional curvature determines the Riemann curvature and vice-versa.

Precisely, suppose V^n $(n \geq 2)$ is a $\mathbb{R}$-inner product space and R and $\tilde{R}$ be two trilinear maps s.t.

$$\langle R(x,y,z), w\rangle \text{ and } \langle \tilde{R}(x,y,z), w\rangle \text{ are}$$

skew in x,y, skew in y,z and satisfy 1st Bianchi identity.

Let $x,y \in V$ be linearly independent.

Let $\sigma = \text{span}\{x,y\}$

Define $$K(\sigma) = \frac{\langle R(x,y,y), x\rangle}{|x \wedge y|^2}$$

$$\tilde{K}(\sigma) = \langle \tilde{R}(x,y,y), x\rangle$$

$$\overline{\qquad\qquad\qquad}$$
$$|x \wedge y|^2$$

if $K = \tilde{K}$ $\forall\ \sigma \subseteq V$ then $R = \tilde{R}$.

<u>Lemma</u> Let V be a real vector space w/ $\dim V \geq 2$ and R and $\tilde{R}$ be trilinear maps $V \times V \times V \to V$ satisfying

$$\langle R(x,y,z), w \rangle = (x,y,z,w)$$
$$\langle \tilde{R}(x,y,z), w \rangle = (x,y,z,w)^{\sim}$$

have the following symmetries :-

$$(x,y,z,w) = -(y,x,z,w) = -(x,y,w,z)$$
$$= (z,w,x,y)$$

and $(x,y,z,w) + (y,z,x,w) + (z,x,y,w) = 0$

same for $\sim$.

Let x, y be linearly independent. Let $\sigma = \operatorname{span}\{x,y\}$

define $\qquad K(\sigma) = \dfrac{(x,y,y,x)}{|x \wedge y|^2}$

$$\tilde{K}(\sigma) = \frac{(x, y, y, x)^{\sim}}{|x \wedge y|^2}$$

If $\tilde{K}(\sigma) = K(\sigma)$ $\forall$ 2-dimensional subspace $\sigma \subseteq V$ then $R = \tilde{R}$.

Proof By hypo. $(x, y, y, x) = (x, y, y, x)^{\sim}$ $\forall x, y$.

polarize $(x+y, z, z, x+y) = (x+y, z, z, x+y)^{\sim}$

$$\Rightarrow (x, z, z, y) + (y, z, z, x) = (x, z, z, y)^{\sim} + (y, z, z, x)^{\sim}$$

$$\Rightarrow 2(x, z, z, y) = 2(x, z, z, y)^{\sim}$$

By symmetries

$$\Rightarrow (x, z, z, y) = (x, z, z, y)^{\sim}$$

polarize again, $z \longmapsto z + w$

$$(x, z, w, y) + (x, w, z, y) = (x, z, w, y)^{\sim} + (x, w, z, y)^{\sim}$$

$$\Rightarrow \underbrace{(x,z,w,y) - (x,z,w,y)^{\sim}} =$$

$$-(x,w,z,y) + (x,w,z,y)^{\sim}$$

$$= (w,x,z,y) - (w,x,z,y)^{\sim}$$

$\Rightarrow$ $\sim$ is invariant under cyclic permutation

$$x \longmapsto z \longmapsto w \longmapsto x$$

$$\Rightarrow \sum_{\substack{x,y,z \\ \text{cyclic}}} (x,z,w,y) - (x,z,w,y)^{\sim}$$

$$= 3(x,z,w,y) - 3(x,z,w,y)^{\sim}$$

$$\Rightarrow 0 - 0 = 0$$

$$\Rightarrow (x,z,w,y) = (x,z,w,y)^{\sim}$$

$\boxed{\text{me}}$

$\underline{2^{nd}\text{ Bianchi Identity}}$

Let (M,g) be Riemannian and R be the

Riemannian (4,0) tensor. Then

$$(\nabla_U R)(X, Y, V, W) + (\nabla_V R)(X, Y, W, U)$$
$$+ (\nabla_W R)(X, Y, U, V) = 0.$$

To prove this, let $p \in M$ be arbitrary and choose Riemannian normal coordinates centred at p. $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$ is a local frame

$$g_{ij}(p) = \delta_{ij}$$

$$\left(\nabla_{\partial_i} \partial_j \right)_p = \Gamma_{ij}^k (p) \, \partial_k \Big|_p = 0$$

Let X, Y, U, V, W be $\partial_i, \partial_j, \partial_k, \partial_l, \partial_m$

now

$$(\nabla_U R)(X, Y, V, W) \underset{\text{defn}}{=} U(R(X, Y, V, W))$$

$$-R(\nabla_U X, Y, V, W)$$

$$- \cdots - R(X, Y, V, \nabla_U W)$$

But $\nabla_U X, \ldots, \nabla_U W = 0$ at p in normal coordinates

$\Rightarrow$ at p, $(\nabla_U R)(X, Y, V, W) = U(R(X, Y, V, W))$

now

$$U(R(X, Y, V, W)) = U(g(R(X,Y)V, W))$$

$$= U(g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \underbrace{\nabla_{[X,Y]}}_{=0 \text{ as coordinate v.f}} V, W))$$

metric compatibility

$$= g(\nabla_U \nabla_X \nabla_Y V, W) - g(\nabla_U \nabla_Y \nabla_X V, W)$$

$$- g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V, \underbrace{\nabla_U W}_{=0 \text{ at } p})$$

$$\Rightarrow (\nabla_U R)(X, Y, V, W)(p) = U(R(X, Y, V, W))(p)$$

$$= U(R(V,W,X,Y),W)(p)$$

$$= g(\nabla_U \nabla_V \nabla_W X, Y)(p)$$
$$- g(\nabla_U \nabla_W \nabla_V X, Y)(p)$$

now cyclically permute U, V and W and then add to get the 2^{nd} Bianchi identity.

$\boxed{\text{///}}$

Remark :- If d^∇ is the exterior covariant derivative then the 2^{nd} Bianchi identity is $d^\nabla R = 0$. (true for any connection on any vector bundle).

Other notions of curvature from Rm

Aside :- Let $(V, \langle , \rangle)$ be an IPS and

$\{e_1, \dots, e_n\}$ be a basis.

$A : V \longmapsto V$ be a linear map.

$Ae_i = A_i^j e_j$. Then $Tr(A) = A_i^i \in \mathbb{R}$.

Notice

$$g^{ij} \langle Ae_i, e_j \rangle = g^{ij} \langle A_i^l e_l, e_j \rangle$$

$$= g^{ij} A_i^l g_{lj} = A_i^i = tr(A)$$

Thus

$$\boxed{tr(A) = g^{ij} \langle Ae_i, e_j \rangle}$$

more generally if B_{ij} is a <u>bilinear form</u>

Define $\quad Tr_g(B) = g^{ij} B_{ij}$.

Let (M, g) Riemannian and fix $X_p, Y_p \in T_pM$

Define $\quad A_p : T_pM \longrightarrow T_pM$ be

$A_p(Z_p) = R(Z_p, X_p) Y_p$

$$\mathrm{Tr}(A_p) = g(A_p e_i, e_j) g_p^{ij}$$

for <u>any</u> basis $e_1, \dots, e_n$ of T_pM.

$$= g(R(e_i, X_p)Y, e_j) g^{ij}$$

$$= R(e_i, X_p, Y_p, e_j) g^{ij}$$

<u>Def^n</u> The <u>Ricci tensor</u> of g is the $(2,0)$ tensor Ric defined

$$\mathrm{Ric}(X,Y) = g^{ij} R(e_i, X, Y, e_j)$$

for any local frame $\{e_1, \dots, e_n\}$

in local coordinates

$$\mathrm{Ric} = R_{jk}\, dx^j \otimes dx^k \quad \text{where}$$

$$R_{jk} = R_{ijkl}\, g^{il}.$$

Remark :- Ricci is symmetric.

Exercise:- Prove the previous remark.

What is the meaning of Ric?

Ric is determined by polarization from its associated quadratic form

$$q(X) = Ric(X,X).$$

Let $\{e_1, \dots, e_n\}$ be a local o-n-frame

$$Ric(e_i, e_i) = g^{kl} R(e_k, e_i, e_i, e_l)$$

$$\overset{\text{o.n.}}{=} \sum_{k=1}^{n} R(e_k, e_i, e_i, e_k)$$

$$= \sum_{k \neq i}^{n} R(e_k, e_i, e_i, e_k)$$

$\overline{}$

$|e_i|^2 |e_k|^2 - \langle e_i, e_k \rangle^2$

$$= \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathcal{K}(e_k \wedge e_i)$$

sectional curvature

2-plane spanned by e_k and e_i

Thus $Ric(e_i, e_i)$ is $(n-1)$ (average of all sectional curvatures of 2-planes containing e_i.)

Scalar Curvature

$$R = Tr_g(Ric) = g^{ij} R_{ij}$$

So R is a smooth function on M.

$$R = n \text{ (average of Ricci curvature)}$$

Special Cases :-

$n=1$: $R_{ijkl}=0$

$n=2$: Ricci , $R_{jk} = g^{il} R_{ijkl}$

$$R_{11} = g^{il} R_{i11l} = g^{22} R_{2112} = g^{22} R_{1221}$$

$$R_{22} = g^{il} R_{i22l} = g^{11} R_{1221} = g^{11} R_{1221}$$

$$R_{12} = g^{il} R_{i12l} = g^{12} R_{2121} = -g^{12} R_{1221}$$

Scalar , $R = g^{11} R_{11} + 2 g^{12} R_{12} + g^{22} R_{22}$

$$= 2\left(g^{11} g^{22} - (g^{12})^2\right) R_{1221}$$

$$= 2 R_{1221} \cdot \det(g^{-1})$$

$$= \frac{1}{\det(g)} 2 R_{1221} = 2K$$

$\therefore$ for $n=2$ $\boxed{R = 2K}$

<u>Defn</u> (M, g) is called <u>Einstein</u> if $\exists$ $\lambda \in C^{\infty}(M)$ s.t

$$\boxed{Ric = \lambda g}$$

Suppose (M, g) is Einstein. Then

$$R = g^{ij} R_{ij} = g^{ij} \lambda g_{ij} = n\lambda$$

$$\Rightarrow \quad \lambda = \frac{R}{n}$$

$$\therefore \quad \boxed{Ric = \frac{R}{n} g}$$

We'll see examples of Einstein metrics.

Special case :- $Ric = 0$ or Ricci-flat.

Aside :- In GR, the natural equation is

$$\underbrace{Ric - \frac{R}{2} g}_{G \,=\, \text{Einstein tensor}} = T - \text{prescribed RHS}$$

↳ stress-energy tensor

Suppose $T = 0 \Rightarrow \quad Ric = R/_2\, g$

tracing $\Rightarrow$

$$R = \frac{nR}{2} \Rightarrow n \neq 2 \Rightarrow$$

$R = 0$ and

$Ric = 0$.

$\therefore$ if $n > 2$ and $T = 0$ then M must be Ricci flat.

Exe. Prove the following :-

① $\nabla_l R_{ljmk} = \nabla_k R_{jm} - \nabla_m R_{jk}$

② $\operatorname{div}(Rc) = \frac{1}{2} dR$.

Lemma :- Diagonalize $\mathcal{R}$ on (M^3, g) w.r.t. basis $\{e_2 \wedge e_3,\ e_3 \wedge e_1,\ e_1 \wedge e_2\}$ of $\Lambda^2 TM^3$ w/ $\{e_1, e_2, e_3\}$ an o.n.b. of TM. Suppose that w.r.t. basis $\mathcal{R}$ is a diagonal matrix w/ entries $\lambda_1, \lambda_2, \lambda_3$. Then w.r.t. $\{e_1, e_2, e_3\}$ we have

$$Rc = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_3 & 0 & 0 \\ 0 & \lambda_3+\lambda_1 & 0 \\ 0 & 0 & \lambda_1+\lambda_2 \end{bmatrix}$$

and the scalar curvature $R = \lambda_1+\lambda_2+\lambda_3$.

Proof. Exercise

Lemma :- Let (M^n, g) be an Einstein manifold w/ $n \geq 3$. Then M has constant scalar curvature. If $n=3$ the g has constant sectional curvature.

Proof. exercise

Defn Constant curvature metrics.

$\mathbb{R}^n$ w/ Euclidean metric has constant sec. curvature 0.

$S^n_R = \{ x \in \mathbb{R}^{n+1}, |x| = R \}$ w/ the round metric has constant sectional curvature $\frac{1}{R^2}$.

$\mathbb{H}^n_R$, the hyperbolic space of radius R which is an open ball of radius R in $\mathbb{R}^n$ w/ the metric

$$g_{ij}(x) = \frac{4R^4 \delta_{ij}}{\quad}$$

$$(R^2 - |x|^2)^2$$

has constant curvature $-1/R^2$.

Any complete, simply connected Riemm. n-fold w/ constant sectional curvature is isometric to one of the above depending on the sign.