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Calibrated Submanifolds

Masterarbeit

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1. Introduction

The notion of calibrations and calibrated submanifolds originates from the seminal paper [HL82] of Harvey and Lawson. Apart from the rich theory of calibrated submanifolds, the link between calibrated geometry and gauge theory (see, e.g., [Lot22] for some examples) has been the reason for a lot of work on special Lagrangian submanifolds in Calabi–Yau manifolds, as well as some on (co-)associative and Cayley submanifolds in G_2- and Spin(7)-manifolds, respectively (see, e.g., [Lot05; Lot06a; Lot06b; Lot12] and the references therein). The core of this thesis lies in constructing special Lagrangian submanifolds of the Calabi–Yau manifold T^*S^n with the Stenzel metric, as well as calibrated submanifolds of the G_2-manifold $\Lambda^2_-(T^*X)$ $(X^4 = S^4, \mathbb{CP}^2)$ and the Spin(7)-manifold $\$_-(S^4)$, both equipped with the Bryant–Salamon metrics.

To begin with, we introduce the notions of calibrations, calibrated submanifolds and holonomy, before discussing the four main examples of calibrated geometries. This includes a short digression on the octonions and their relation to the Lie groups G_2 and Spin(7), which we use to collect different characterizations of (co-)associative and Cayley submanifolds established in [HL82; KM05]. A brief insight into the classification of calibrations concludes Section 2.

The third section gives a review of previous works that motivate this thesis, and is used to fix our setup and notation. Inspired by the Harvey–Lawson bundle construction of special Lagrangian submanifolds in $\mathbb{C}^n$ [HL82], Ionel–Karigiannis–Min-Oo [IKM05] described similar constructions of (co-)associative submanifolds in $\mathbb{R}^7$ and Cayley submanifolds in $\mathbb{R}^8$. The idea is to view the ambient manifold as the total space of a vector bundle over some Euclidean space $\mathbb{R}^n$, restricting it to an oriented immersed submanifold $L \subset \mathbb{R}^n$ and then considering the total spaces of appropriate subbundles. More precisely, Harvey–Lawson [HL82] viewed $\mathbb{C}^n \cong T^*\mathbb{R}^n$ as the cotangent bundle and then considered the conormal bundle N^*L^q. Similarly, Ionel–Karigiannis–Min-Oo [IKM05] viewed $\mathbb{R}^7 \cong \Lambda^2_-(T^*\mathbb{R}^4)$ as the space of anti-self-dual 2-forms on $\mathbb{R}^4$ and $\mathbb{R}^8 \cong \$_-(\mathbb{R}^4)$ as the negative spinor bundle of $\mathbb{R}^4$. They examined naturally defined subbundles E and $F = E^\perp$ of $\Lambda^2_-(T^*\mathbb{R}^4)|_{L^2}$ of rank 1 and 2, and V_+ and $V_- = V_+^\perp$ of $\$_-(\mathbb{R}^4)|_{L^2}$ of rank 2. Later, Karigiannis–Min-Oo [KM05] generalized these constructions to complete, nonflat, noncompact manifolds of special holonomy which are total spaces of vector bundles over a compact base. In other words, they examined the analogs of these submanifolds in the Calabi–Yau manifold T^*S^n with the Stenzel metric, in $\Lambda^2_-(T^*X)$ $(X^4 = S^4, \mathbb{CP}^2)$ with the Bryant–Salamon metric of holonomy G_2 and in $\$_-(S^4)$ with the Bryant–Salamon metric of holonomy Spin(7). The authors of [HL82], [IKM05] and [KM05] proved the following results: First, the conormal bundle N^*L is special Lagrangian in T^*X if and only if L^q is austere in $X^n = \mathbb{R}^n, S^n$. Second, the submanifold E (F) is associative (coassociative) in $\Lambda^2_-(T^*X)$ if and only if L^2 is minimal (negative superminimal) in $X^4 = \mathbb{R}^4, S^4, \mathbb{CP}^2$. Third, the submanifold $V_\pm$ is Cayley in $\$_-(X)$ if and only if L^2 is minimal in $X^4 = \mathbb{R}^4, S^4$. Inspired by Borisenko [Bor93], Karigiannis–Leung [KL12] further generalized [IKM05] by "twisting" the subbundles by special sections of the complementary bundles. They derived conditions on L and the sections in order to obtain calibrated submanifolds of the Euclidean spaces $\mathbb{C}^n \cong T^*\mathbb{R}^n$, $\mathbb{R}^7 \cong \Lambda^2_-(T^*\mathbb{R}^4)$ and $\mathbb{R}^8 \cong \$_-(\mathbb{R}^4)$.

Section 4 represents the core of this thesis: We generalize the constructions in [KL12] to complete, nonflat, noncompact manifolds of special holonomy. In other words, we twist the calibrated subbundles in T^*S^n, $\Lambda^2_-(T^*X)$ ($X^4 = S^4, \mathbb{CP}^2$) and $\$_-(S^4)$ constructed in [KM05] by special sections. Our main results are contained in Theorem 4.1, Theorem 4.4 and Theorem 4.7. In Theorem 4.1, we find that twisting the conormal bundle N^*L by a 1-form $\mu \in \Omega^1(L^q)$ does not provide any new examples because the Lagrangian condition requires μ to vanish. This differs from the case of $\mathbb{R}^4$ in [KL12], where the authors found that the twisted conormal bundle is special Lagrangian in $T^*\mathbb{R}^n$ if and only if μ is closed and its symmetrized covariant derivative satisfies certain equations. Theorem 4.4 describes the (co-)associative case: We show that the bundle E twisted by a section $\sigma \in \Gamma(F)$ is associative in $\Lambda^2_-(T^*X)$ if and only if L^2 is minimal in $X^4 = S^4, \mathbb{CP}^2$ and σ is holomorphic. On the other hand, the complementary bundle F twisted by a section $\eta \in \Gamma(E)$ is coassociative if and only if L^2 is negative superminimal in X^4 and η is parallel. Lastly, Theorem 4.7 proves that the bundle V_+ twisted by a section $\psi \in \Gamma(V_-)$ is Cayley in $\$_-(S^4)$ if and only if L^2 is minimal in S^4 and ψ is holomorphic. The conditions on L^2 and the sections $\sigma \in \Gamma(F)$, $\eta \in \Gamma(E)$ and $\psi \in \Gamma(V_-)$ for the G_2-manifold $\Lambda^2_-(T^*X)$ ($X^4 = S^4, \mathbb{CP}^2$) and the Spin(7)-manifold $\$_-(S^4)$ turn out to be the same as in the case of $\mathbb{R}^4$ in [KL12]. However, in contrast to [KL12], none of the presented proofs rely on identifications with the (purely imaginary) octonions. Instead, they are based on the vanishing of certain (bundle-valued) differential forms, as established in [HL82; KM05].

Our findings demonstrate that the constructions of calibrated submanifolds in Euclidean spaces in [KL12] cannot be entirely extended to the manifolds T^*S^n, $\Lambda^2_-(T^*X)$ ($X^4 = S^4, \mathbb{CP}^2$) and $\$_-(S^4)$ considered in [KM05]. While the results for the two spaces of exceptional holonomy are in line with the previous findings, the construction in T^*S^n does not provide any new examples. As in [KL12], the (co-)associative and Cayley subbundles constructed in [KM05] allow deformations destroying the linear structure of the fiber, while the base space L^2 remains of the same type after twisting, namely minimal or negative superminimal. This implies that the moduli space of calibrated submanifolds near a calibrated subbundle of this kind not only contains deformations of the base L but also of the fiber. In contrast, the special Lagrangian bundle construction in T^*S^n is much more rigid than in the case of $T^*\mathbb{R}^n$. Closing this final section, we point out potential future studies on the existence of other types of deformations in the above three cases and the possibility of finding analogous results for other manifolds of special holonomy.

Finally, Appendix A contains a tedious computation omitted in Section 4, Appendix B gives a review of spin geometry which provides additional background for the Cayley construction, and Appendix C presents the octonion multiplication table.

2. Calibrated geometry

The purpose of this second section is to review calibrated geometry. This includes a short motivation via minimal submanifolds, a brief introduction to holonomy, and the four main examples of calibrated geometries. Additionally, we make a quick digression on the octonions and their relation to the groups G_2 and $\mathrm{Spin}(7)$, which we use to collect different characterizations of (co-)associative and Cayley submanifolds derived in [HL82; KM05].

2.1. Minimal and calibrated submanifolds

Let us first address submanifolds of smooth manifolds.

Definition 2.1 ([Wen22, Sec. 4], [Joy07, Def. 4.1.1])**.** Let M, N be smooth manifolds and $f : N \to M$ be a smooth map. We call f an **immersion** if the map $D_p f : T_p N \to T_{f(p)} M$ is injective for every $p \in N$. In that case, we say that N (or $f(N) \subset M$) is an **immersed submanifold** of M. If f is additionally injective with continuous inverse $f^{-1} : f(N) \to N$, we call it an **embedding** and N (or $f(N) \subset M$) an **embedded submanifold** of M. Two immersed submanifolds $f : N \to M$ and $f' : N' \to M$ are **isomorphic** if there exists a diffeomorphism $\Psi : N \to N'$ such that $f = f' \circ \Psi$. If this is the case, we consider them to be the same.

As for subsets L of M, we say that L is a **(smooth) submanifold** of M if it admits a smooth structure such that the inclusion map $L \hookrightarrow M$ is an embedding. It is common to implicitly identify an immersed submanifold N of M with its image $f(N) \subset M$ and to not mention f at all. This is reasonable for embedded submanifolds because $f(N)$ is a smooth submanifold of M [Wen22, Thm. 4.14] and the inclusion $f(N) \hookrightarrow M$ is isomorphic to $f : N \to M$ as $N \xrightarrow{f} f(N)$ is a diffeomorphism. In particular, this shows that the terms *(smooth) submanifold* and *embedded submanifold* are interchangeable. When dealing with non-embedded immersed submanifolds, however, caution is required. If f has self-intersections $f(p) = f(q) \in M$ for $p \neq q$, there are two possible scenarios: either $f(N)$ has singularities, that is, $f(N)$ is not a submanifold of M, or $f(N)$ is a nonsingular submanifold of M with nontrivial multiple cover $N \xrightarrow{f} f(N)$, which makes it impossible to reconstruct N and f up to isomorphism from $f(N)$. As an immersion is a local embedding [Wen22, Thm. 4.11], we follow this convention nonetheless, but take it with a grain of salt. (See [Wen22, Sec. 4] for more details.)

From now on, let (M, g) be a Riemannian manifold of dimension n and $1 \leq k \leq n-1$.

Definition 2.2 ([Lot22, Sec. 2.1])**.** Let N be an oriented immersed submanifold of M with immersion $f : N \to M$. A **variation** of N **with compact support** is a smooth one-parameter family of immersions $\{f_t : N \to M\}_{t \in (-\varepsilon, \varepsilon)}$ for which there exists an open set $S \subset N$ with compact closure $\overline{S}$ such that $f_0 = f$ and $f_t|_{N \setminus \overline{S}} = f|_{N \setminus \overline{S}}$ for all $t \in (-\varepsilon, \varepsilon)$. We call N **minimal** if $\frac{d}{dt} \mathrm{Vol}(f_t(S))|_{t=0} = 0$ holds for all variations $\{f_t\}_{t \in (-\varepsilon, \varepsilon)}$ with compact support $\overline{S}$ (depending on the variation).

Minimal submanifolds are characterized by a second-order nonlinear partial differential equation on the immersion map of the submanifold (see also Remark 3.2) and are therefore difficult to analyze. Additionally, N being minimal does not necessarily mean that it minimizes volume. The condition only requires N to be a critical point of the volume functional, which is even satisfied by volume-maximizing submanifolds. These two issues can be resolved by working with calibrated submanifolds instead, which were introduced by [HL82].

Definition 2.3 ([Joy07, Def. 4.1.3], [Lot22, Sec. 2.3], [KL12, Sec. 1]). An **oriented tangent k-plane** on M is an oriented k-dimensional vector subspace V of some tangent space T_pM to M. Given such a V, $g|_V$ together with the orientation on V gives a natural volume form on V, which we denote by $\mathrm{vol}_V \in \Lambda^k(V^*)$.

A **k-calibration** on M is a closed k-form $\varphi \in \Omega^k(M)$ which satisfies $\varphi|_V \leq \mathrm{vol}_V$ for all oriented tangent k-planes V on M. That is, $d\varphi = 0$ and $\varphi(e_1, \ldots, e_k) \leq 1$ for all orthonormal tangent vectors $e_1, \ldots, e_k$ to M. We call a k-dimensional oriented immersed submanifold N of M **calibrated** by φ if $\varphi|_{T_pN} = \mathrm{vol}_{T_pN}$ for all $p \in N$ or, equivalently, if for all $p \in N$, $\varphi(e_1, \ldots, e_k) = 1$ for an oriented orthonormal basis $e_1, \ldots, e_k$ for T_pN.

If M is the total space of a vector bundle over a base X and a calibrated submanifold N is also the total space of a subbundle, we call N a **calibrated subbundle** of M. In this context, a **subbundle** of $M \to X$ means a vector bundle $N \to L$ over a submanifold L of X, whose fibers are subspaces of the corresponding fibers of M.

Before comparing this notion to minimal submanifolds, let us prove a lemma that will play an important role when we look at the main examples of calibrated submanifolds and the classification of calibrations. Whenever (M^n, g) is equipped with an orientation, the Hodge star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ provides a natural one-to-one correspondence between its k-forms and $(n-k)$-forms. The **Hodge dual** $*\beta$ of some $\beta \in \Omega^k(M)$ is defined as the unique $(n-k)$-form satisfying $\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \, \mathrm{vol}_M$ for all $\alpha \in \Omega^k(M)$. Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\Omega^k(M)$ induced by g (see (B.2)) and vol_M stands for the natural volume form on (M, g) determined by g and the orientation on M. The operator is well-defined and depends on g and the orientation. (See [Joy07, Ch. 1.1.2] for more details.) Now suppose $e_1, \ldots, e_n$ is an oriented local orthonormal frame with dual coframe $e^1, \ldots, e^n$ and let $\beta \in \Omega^k(M)$. Locally, we have

$$\langle \beta, \beta \rangle \, \mathrm{vol}_M = \sum_{\sigma \in \mathrm{Sh}_{k,n-k}} \beta(e_{\sigma(1)}, \ldots, e_{\sigma(k)})^2 \, e^1 \wedge \cdots \wedge e^n$$

and

$$\beta \wedge (*\beta) = \sum_{\sigma \in \mathrm{Sh}_{k,n-k}} (-1)^{|\sigma|} \beta(e_{\sigma(1)}, \ldots, e_{\sigma(k)}) (*\beta)(e_{\sigma(k+1)}, \ldots, e_{\sigma(n)}) \, e^1 \wedge \cdots \wedge e^n,$$

where $\mathrm{Sh}_{k,n-k} = \{\sigma \in S_n \mid \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(n)\}$ is the set of $(k, n-k)$-shuffles. Comparing these formulas, we see that the Hodge dual satisfies

$$(*\beta)(e_{\sigma(k+1)}, \ldots, e_{\sigma(n)}) = (-1)^{|\sigma|} \beta(e_{\sigma(1)}, \ldots, e_{\sigma(k)}). \tag{2.1}$$

Lemma 2.4 ([Lot22, Lemma 3.4]). *Let (M^n, g) be an oriented Riemannian manifold and $\varphi \in \Omega^k(M)$ be a calibration whose Hodge dual $*\varphi$ is closed. Then $*\varphi \in \Omega^{n-k}(M)$ is a calibration too. Moreover, an oriented tangent k-plane $V \subset T_pM$ is calibrated by φ if and only if we can equip its orthogonal complement $V^\perp \subset T_pM$ with an orientation so that it is calibrated by $*\varphi$.*

Proof. Let $\varphi \in \Omega^k(M)$ be a calibration with $d(*\varphi) = 0$. Fix a point $p \in M$ and consider $n-k$ orthonormal tangent vectors $e_{k+1}, \dots, e_n$ at p. We can find $e_1, \dots, e_k \in T_pM$ such that $(e_1, \dots, e_n)$ is an oriented orthonormal basis of T_pM. Since $(e_1, \dots, e_n)$ is positively oriented and φ is a calibration, (2.1) shows that

$$(*\varphi)(e_{k+1}, \dots, e_n) = \varphi(e_1, \dots, e_k) \leq 1.$$

Thus, $*\varphi \in \Omega^{n-k}(M)$ is a calibration.

Suppose $V \subset T_pM$ is an oriented tangent k-plane with oriented orthonormal basis $e_1, \dots, e_k \in T_pM$ and its orthogonal complement $V^\perp$ is spanned by orthonormal tangent vectors $e_{k+1}, \dots, e_n \in T_pM$. Then the terms $\varphi(e_1, \dots, e_k)$ and $(*\varphi)(e_{k+1}, \dots, e_n)$ can only differ by a sign and are equal whenever $(e_1, \dots, e_n)$ is positively oriented. Hence, V is calibrated by φ if and only if we have $(*\varphi)(e_{k+1}, \dots, e_n) = \pm 1$. After changing the orientation on $V^\perp$ if necessary, the second condition is equivalent to $V^\perp$ being calibrated by $*\varphi$. □

Finally, we observe that calibrated submanifolds offer two key advantages over minimal submanifolds: First, calibrated submanifolds are characterized by an algebraic condition on the tangent vectors to N, which translates into a nonlinear partial differential equation of only first-order on the immersion map. Second, they are always volume-minimizing in the following sense:

Theorem 2.5 ([Lot22, Thm. 2.7])**.** *Let N be a calibrated submanifold of M with immersion $f : N \to M$. Then N is minimal and volume-minimizing in the sense that $\mathrm{Vol}(S) \leq \mathrm{Vol}(f_t(S))$ for all variations $\{f_t : N \to M\}_{t \in (-\varepsilon, \varepsilon)}$ of N with compact support $\overline{S}$. If N is additionally compact, it is volume-minimizing in its homology class.*

Proof. Suppose N is calibrated by $\varphi \in \Omega^k(M)$ and $\{f_t : N \to M\}_{t \in (-\varepsilon, \varepsilon)}$ is a variation of the immersion $f : N \to M$ with compact support $\overline{S}$. Then the natural volume form on N is given by $\mathrm{vol}_N = f^*\varphi \in \Omega^k(N)$. As f is an immersion, it is a local embedding. Hence, we can find a partition of unity $\{(U_\alpha, \psi_\alpha)\}_\alpha$ on S such that $f|_{U_\alpha} : U_\alpha \to f(U_\alpha)$ is a diffeomorphism for all α. We compute

$$\mathrm{Vol}(S) = \int_S \mathrm{vol}_N = \int_S f^*\varphi = \sum_\alpha \int_{U_\alpha} \psi_\alpha f^*\varphi = \sum_\alpha \int_{f(U_\alpha)} (\psi_\alpha \circ (f|_{U_\alpha})^{-1})\varphi = \int_{f(S)} \varphi.$$

As φ is closed and $f_t|_{N \setminus \overline{S}} = f|_{N \setminus \overline{S}}$, Stokes' theorem gives

$$\int_{f(S)} \varphi - \int_{f_t(S)} \varphi = \int_{\partial K} \varphi = \int_K d\varphi = 0,$$

where $K \subset M$ is the compact set enclosed by $f(S)$ and $f_t(S)$. Combining this, we obtain

$$\mathrm{Vol}(S) = \int_{f(S)} \varphi = \int_{f_t(S)} \varphi \leq \int_{f_t(S)} \mathrm{vol}_{f_t(S)} = \mathrm{Vol}\big(f_t(S)\big).$$

In particular, N must be minimal.

Now suppose N is compact and N' is homologous to N. Then

$$\mathrm{Vol}(N) = \int_N \mathrm{vol}_N = \int_N \varphi = [\varphi] \cdot [N] = [\varphi] \cdot [N'] = \int_{N'} \varphi \leq \int_{N'} \mathrm{vol}_{N'} = \mathrm{Vol}(N'),$$

where $[\varphi] \cdot [N]$ stands for the pairing between the cohomology class of φ in $H^k(M)$ and the homology class of N in $H_k(M)$ (see [Lee03, Ch. 16] for more details). □

2.2. Calibrated geometry and holonomy

Before we come to the main examples of calibrations and calibrated submanifolds, let us first introduce a notion which can serve as a hint for the existence of calibrated submanifolds, called *holonomy*.

Definition 2.6 ([Kar20, Def. 5.1])**.** Let (M^n, g) be a Riemannian manifold with Levi-Civita connection ∇ and fix a point $p \in M$. A **loop** based at p is a continuous and piecewise smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p$. Given such a loop, the parallel transport map $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ along γ with respect to ∇ lies in the group of isometries $\mathrm{O}(T_pM)$ as $\nabla g = 0$. We define the **holonomy group** $\mathrm{Hol}_p(g)$ of g at p to be

$$\mathrm{Hol}_p(g) = \{P_\gamma : T_pM \to T_pM \mid \gamma \text{ is a loop based at } p\} \subset \mathrm{O}(T_pM).$$

By the existence and uniqueness of parallel transport, we have $P_\gamma P_\delta = P_{\gamma\delta}$ and $P_\gamma^{-1} = P_{\gamma^{-1}}$ for all loops γ, δ based at p, which proves that $\mathrm{Hol}_p(g)$ is indeed a subgroup of $\mathrm{O}(T_pM)$. After fixing an isomorphism $T_pM \cong \mathbb{R}^n$, we can view $\mathrm{Hol}_p(g)$ as a subgroup of the orthogonal group $\mathrm{O}(n)$. In fact, the conjugacy class of that subgroup is independent of the choice of isomorphism, which justifies switching between viewing $\mathrm{Hol}_p(g)$ as a subgroup of $\mathrm{O}(T_pM)$ and $\mathrm{O}(n)$ without choosing an explicit isomorphism first. Moreover, we have $\mathrm{Hol}_p(g) \cong \mathrm{Hol}_q(g)$ whenever $p, q \in M$ lie in the same connected component of M. Combining this, we see that for every connected component C_α of M, there exists a unique (up to conjugation) subgroup H_α of $\mathrm{O}(n)$ such that $\mathrm{Hol}_p(g) \cong H_\alpha$ for all $p \in C_\alpha$. (See [Kar20, Prop. 5.2].) We say that (M, g) has **holonomy** in (equal to) a subgroup G of $\mathrm{O}(n)$ if we have $H_\alpha \subset G$ ($H_\alpha = G$) for every α. Furthermore, we refer to the smallest group $G \subset \mathrm{O}(n)$ fulfilling this condition as the **holonomy group** of (M, g) and denote it by $\mathrm{Hol}(g)$. Whenever $\mathrm{Hol}(g)$ is a proper subgroup of $\mathrm{O}(n)$ (or $\mathrm{SO}(n)$ if M is orientable), we call (M, g) a manifold of **reduced** or **special holonomy**.

This concept obeys the following central principle.

Proposition 2.7 (Holonomy principle, [Joy07, Prop. 2.5.2])**.** *Let (M, g) be a connected Riemannian manifold with Levi-Civita connection ∇ and fix a point $p \in M$. Then we obtain a natural connection ∇ on the vector bundle $E = (TM)^{\otimes s} \otimes (T^*M)^{\otimes t}$ and a natural action of $\mathrm{Hol}_p(g)$ on its fiber E_p via the pullback.*

If $S \in \Gamma(E)$ is a parallel (s,t)-tensor, then $\mathrm{Hol}_p(g)$ leaves $S|_p$ invariant. Conversely, if $\mathrm{Hol}_p(g)$ leaves $S_p \in E_p$ invariant, there exists a unique parallel tensor $S \in \Gamma(E)$ that satisfies $S|_p = S_p$.

Proof. Suppose $S \in \Gamma(E)$ is a parallel (s,t)-tensor. That is, we have $P_\alpha^*(S|_{\alpha(1)}) = S|_{\alpha(0)}$ for every piecewise smooth path $\alpha : [0,1] \to M$. In particular, S satisfies $P_\gamma^*(S|_p) = S|_p$ for every loop γ based at p, which proves that $S|_p$ is invariant under $\mathrm{Hol}_p(g)$.

Conversely, suppose $S_p \in E_p$ is fixed by $\mathrm{Hol}_p(g)$. Let $q \in M$ be any point. As M is connected, we can find piecewise smooth paths $\alpha, \beta : [0,1] \to M$ with $\alpha(0) = \beta(0) = q$ and $\alpha(1) = \beta(1) = p$. Then $\alpha\beta^{-1}$ is a loop based at p and therefore $P_{\alpha\beta^{-1}} = P_\alpha P_\beta^{-1} \in \mathrm{Hol}_p(g)$. Thus, we have

$$P_\alpha^*(S_p) = (P_\alpha P_\beta^{-1} P_\beta)^*(S_p) = P_\beta^*\big((P_\alpha P_\beta^{-1})^*(S_p)\big) = P_\beta^*(S_p),$$

which proves that $P_\alpha^*(S_p) \in E_q$ depends only on q and is independent of the choice of α. Due to this, we can define the tensor $S \in \Gamma(E)$ by $S|_q = P_\alpha^*(S_p) \in E_q$, where α is any piecewise smooth path from q to p. By definition, this is the unique parallel tensor satisfying $S|_p = S_p$. □

Remark 2.8. Throughout this thesis, we always work with this natural connection ∇ on E, unless stated otherwise.

We can use this concept to develop a promising approach for finding calibrated submanifolds: To begin with, let us assume that (M, g) is connected and pick a point $p \in M$. Suppose $\varphi_p \in \Lambda^k(T_p^*M)$ is nonzero and $\mathrm{Hol}_p(g)$-invariant. Then we can rescale φ_p such that $\varphi_p|_V \leq \mathrm{vol}_V$ holds for all oriented tangent k-planes $V \subset T_pM$ at p with equality for at least one of them. Whenever $V \subset T_pM$ is calibrated by φ_p, so is $(P_\gamma)^*V$ for all $P_\gamma \in \mathrm{Hol}_p(g)$ because φ_p is $\mathrm{Hol}_p(g)$-invariant. In most cases, this means that a variety of such calibrated planes at p exists. The holonomy principle now provides us with a unique parallel k-form $\varphi \in \Omega^k(M)$ satisfying $\varphi|_p = \varphi_p$. As $\nabla\varphi = 0$, φ is closed. Moreover, since φ and g are parallel, the condition $\varphi_p|_V \leq \mathrm{vol}_V$ at p implies $\varphi|_V \leq \mathrm{vol}_V$ for every oriented tangent k-plane V at any point in M. This proves that $\varphi \in \Omega^k(M)$ is a calibration. Additionally, the invariance of φ under $\mathrm{Hol}(g)$ promises a large number of calibrated planes at any point in M, which allows us to hope for calibrated submanifolds. (See [Joy07, Ch. 4.2] for more details.) For manifolds (M, g) with multiple connected components C_α, we simply follow this approach on each C_α and then piece together the resulting calibrations $\varphi_\alpha \in \Omega^k(C_\alpha)$ to one calibration $\varphi \in \Omega^k(M)$ on M.

2.3. Main examples and equivalent criteria

In the following, we use the idea presented in the previous subsection to construct interesting calibrations on manifolds of reduced holonomy, for which the existence of calibrated submanifolds seems likely. We discuss the four main examples of calibrated geometries and characterize the corresponding calibrated submanifolds. In preparation for the last two examples, we additionally include a brief digression on the octonions and their relation to the groups G_2 and $\mathrm{Spin}(7)$.

2.3.1. Kähler manifolds and complex submanifolds

We start by introducing *Kähler manifolds*. To begin with, we give a rather hands-on definition, before deriving an alternative characterization via holonomy.

Definition 2.9 (Kähler manifold I, [Joy07, Ch. 5.4])**.** Let (M, J) be a complex manifold with Riemannian metric g. We call g **Hermitian** if it satisfies $g(u, v) = g(Ju, Jv)$ for all vector fields $u, v \in \Gamma(TM)$. In that case, its **associated 2-form** ω is defined as $\omega(u, v) = g(Ju, v)$, $u, v \in \Gamma(TM)$. If $d\omega = 0$, we call g a **Kähler metric**, ω the **Kähler form** and the quadruple (M, g, J, ω) a **Kähler manifold**.

Example 2.10 ([Lot22, Sec. 3.1], [Huy05, p. 42]). Consider $\mathbb{C}^n$ with the standard coordinates $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and the natural complex structure $\widetilde{J}$ defined via

$$\widetilde{J}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad \widetilde{J}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

The standard metric

$$\widetilde{g} = \sum_{j=1}^{n}(dx_j \otimes dx_j + dy_j \otimes dy_j) = \mathrm{Re}\Big(\sum_{j=1}^{n} dz_j \otimes d\bar{z}_j\Big)$$

on $(\mathbb{C}^n, \widetilde{J})$ is Hermitian as $dx_j \circ \widetilde{J} = -dy_j$ and $dy_j \circ \widetilde{J} = dx_j$, $j = 1, \dots, n$. Furthermore, its associated 2-form $\widetilde{\omega}$ is given by

$$\widetilde{\omega} = \sum_{j=1}^{n} dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

and is clearly closed. As a result, $(\mathbb{C}^n, \widetilde{g}, \widetilde{J}, \widetilde{\omega})$ is a Kähler manifold.

Let (M, J) be a complex manifold of complex dimension n with Hermitian metric g, associated 2-form ω (not necessarily closed) and Levi-Civita connection ∇. Using that g is Hermitian and that ∇ is compatible with g, one shows that

$$(\nabla_u \omega)(v, w) = g\big((\nabla_u J)(v), w\big) = -g\big((\nabla_u J)(w), v\big)$$

and thus,

$$d\omega(u, v, w) = g\big((\nabla_u J)(v), w\big) + g\big((\nabla_v J)(w), u\big) + g\big((\nabla_w J)(u), v\big)$$

for $u, v, w \in \Gamma(TM)$. On the other hand, the Koszul formula yields

$$2g\big((\nabla_u J)(v), w\big) = d\omega(u, v, w) - d\omega(u, Jv, Jw)$$

[Bal06, Prop. 4.16]. From this, we deduce that $d\omega = 0 \iff \nabla\omega = 0 \iff \nabla J = 0$. Now suppose ω is closed. Then $J|_p = P_\gamma^*(J|_p) = P_\gamma^{-1} \circ J|_p \circ P_\gamma$ and $\omega|_p = P_\gamma^*(\omega|_p) = \omega|_p(P_\gamma \cdot, P_\gamma \cdot)$ hold for any loop γ based at $p \in M$. In other words, $J|_p$ and $\omega|_p$ are invariant under $\mathrm{Hol}_p(g)$ (cf. Proposition 2.7). The first condition is equivalent to demanding that P_γ is complex linear and therefore requires $\mathrm{Hol}_p(g)$ to lie in the group of unitary matrices $\mathrm{U}(n)$. The second condition already follows from the first one and the fact that g is parallel:

$$\omega|_p(P_\gamma u, P_\gamma v) = g|_p\big(J|_p(P_\gamma u), P_\gamma v\big) = g|_p\big(P_\gamma(J|_p u), P_\gamma v\big) = g|_p(J|_p u, v) = \omega|_p(u, v)$$

for $u, v \in T_pM$. Thus, a Kähler manifold has holonomy in $\mathrm{U}(n)$.

Conversely, let us start with a Riemannian manifold (M, g) of dimension $2n$ with holonomy $\mathrm{Hol}(g) \subset \mathrm{U}(n)$ and fix a point $p \in M$. This means that there exists a complex structure J_p at p such that $g|_p$ is Hermitian and $\mathrm{Hol}_p(g) \subset \mathrm{U}(n)$ with respect to J_p. Due to this, we can identify T_pM with $\mathbb{C}^n$, which provides us with a Kähler form ω_p corresponding to the standard Kähler form $\widetilde{\omega}$ on $\mathbb{C}^n$ (see Example 2.10). Since every $A \in \mathrm{Hol}_p(g) \subset \mathrm{U}(n)$ is complex linear, we have $AJ_pA^{-1} = J_p$ and, consequently, also

$$\omega_p(Au, Av) = g|_p\big(J_p(Au), Av\big) = g|_p\big(A(J_p u), Av\big) = g|_p(J_p u, v) = \omega_p(u, v)$$

for $u, v \in T_pM$, where we used that $g|_p$ is Hermitian. Given that J_p and ω_p are invariant under $\mathrm{Hol}_p(g)$, extending them via parallel transport yields well-defined parallel tensors J_α and ω_α on the connected component C_α of M containing p (cf. Proposition 2.7). Carrying out this construction on every connected component and then piecing together the resulting tensors leads to global parallel tensors J and ω. Since g, J and ω are parallel, their compatibility and the condition $J^2 = -\mathrm{Id}$ are maintained. Furthermore, $\nabla J = 0$ shows that J indeed gives a complex structure on M and $\nabla\omega = 0$ implies that $d\omega = 0$. Combining all of this, we see that (M, g, J, ω) is a Kähler manifold according to Definition 2.9. This leads to the following equivalent definition.

Definition 2.11 (Kähler manifold II, [Lot22, Def. 3.1])**.** A **Kähler manifold** is a Riemannian manifold (M, g) of dimension $2n$ with $\mathrm{Hol}(g) \subset \mathrm{U}(n)$.

Let (M^{2n}, g) be a Kähler manifold. As discussed in the previous subsection, the above characterization gives hope for finding interesting calibrations and, consequently, calibrated submanifolds of M. At a point $p \in M$, we consider the Kähler form $\omega_p \in \Lambda^2(T_p^*M)$ derived from $\widetilde{\omega}$ on $\mathbb{C}^n$, which, as we saw above, is invariant under $\mathrm{U}(n)$. Wirtinger's inequality [Lot22, Thm. 3.3] shows that for any orthonormal vectors $e_1, \ldots, e_{2k} \in \mathbb{C}^n$, we have

$$\frac{\widetilde{\omega}^k}{k!}(e_1, \ldots, e_{2k}) \leq 1$$

with equality if and only if the vectors $e_1, \ldots, e_{2k}$ span a complex k-plane in $\mathbb{C}^n$, that is, $\widetilde{J}(\mathrm{span}\{e_1, \ldots, e_{2k}\}) = \mathrm{span}\{e_1, \ldots, e_{2k}\}$. Extending ω_p parallelly to $\omega \in \Omega^2(M)$ not only yields $d\omega = 0$ and therefore

$$d\left(\frac{\omega^k}{k!}\right) = \frac{k\, d\omega \wedge \omega^{k-1}}{k!} = \frac{d\omega \wedge \omega^{k-1}}{(k-1)!} = 0,$$

but also preserves the condition $\frac{\omega^k}{k!}|_V \leq \mathrm{vol}_V$ for every oriented tangent $2k$-plane V on M. Thus, we obtain the following result.

Theorem 2.12 ([Lot22, Thm. 3.2])**.** *A Kähler manifold (M, g, J, ω) possesses natural calibrations given by $\frac{\omega^k}{k!}$, whose calibrated submanifolds are the complex k-dimensional submanifolds, i.e., those submanifolds $N \subset M$ satisfying $J(T_pN) = T_pN$ for every $p \in N$.*

2.3.2. Calabi–Yau manifolds and special Lagrangian submanifolds

Let us move on to *Calabi–Yau manifolds.* As before, we give two equivalent definitions: one based on the existence of a special differential form and the other in terms of holonomy.

Definition 2.13 (Calabi–Yau manifold I; [GJH03, Prop. 4.5], [Joy07, Def. 7.1.10])**.** A **Calabi–Yau manifold** is a Ricci-flat Kähler manifold (M, g, J, ω) of complex dimension n which admits a nowhere vanishing holomorphic $(n, 0)$-form Ω, called a **holomorphic volume form**. We write $(M, g, J, \omega, \Omega)$ for the given data.

Remark 2.14. There are several different definitions of *Calabi–Yau manifolds* in the literature. While most of them require the manifold to be compact, we use a broader definition, which allows us to equip the cotangent bundles $T^*\mathbb{R}^n$ and T^*S^n with Calabi–Yau structures (see Subsection 3.2). Whenever the manifold is compact, the existence of a holomorphic volume form already guarantees that there exists a Ricci-flat Kähler metric on (M, J) [Joy07, Thm. 7.1.2].

Example 2.15 ([Lot22, Sec. 3.2]). Let us revisit the Kähler manifold $(\mathbb{C}^n, \widetilde{g}, \widetilde{J}, \widetilde{\omega})$ from Example 2.10. Its natural holomorphic volume form is given by

$$\widetilde{\Omega} = dz_1 \wedge \cdots \wedge dz_n.$$

As $\widetilde{g}$ is (Ricci-)flat, this turns $(\mathbb{C}^n, \widetilde{g}, \widetilde{J}, \widetilde{\omega}, \widetilde{\Omega})$ into a Calabi–Yau manifold.

According to [GJH03, Prop. 4.5], a Calabi–Yau manifold has holonomy in the group of special unitary matrices $\mathrm{SU}(n)$. Conversely, suppose that a Riemannian manifold (M^{2n}, g) has holonomy in $\mathrm{SU}(n)$. As $\mathrm{SU}(n)$ lies in $\mathrm{U}(n)$, this is simply a Kähler manifold (M, g, J, ω) with $\mathrm{Hol}(g) \subset \mathrm{SU}(n)$. By [GJH03, Prop. 4.5], g is a Ricci-flat. Now fix a point $p \in M$. Identifying T_pM with $\mathbb{C}^n$ provides us with a nonzero $(n,0)$-form Ω_p corresponding to the standard holomorphic volume form $\widetilde{\Omega}$ on $\mathbb{C}^n$ (see Example 2.15). Since every $A \in \mathrm{Hol}_p(g) \subset \mathrm{SU}(n)$ has determinant 1, we have $A^*\Omega_p = (\det A)\,\Omega_p = \Omega_p$. Given that Ω_p is invariant under $\mathrm{Hol}_p(g)$, extending it via parallel transport yields a well-defined parallel $(n,0)$-form Ω_α on the connected component C_α of M containing p (cf. Proposition 2.7). Carrying out this construction on every connected component and then piecing together the resulting forms leads to a global parallel $(n,0)$-form Ω. Since Ω is parallel and Ω_p nonzero, Ω vanishes nowhere. Furthermore, Ω being parallel implies that it is closed, and since Ω is an $(n,0)$-form, this is equivalent to Ω being holomorphic [Huy05, Lemma 1.3.6]. Thus, we see that $(M, g, J, \omega, \Omega)$ is a Calabi–Yau manifold according to Definition 2.13. This leads to the following equivalent definition.

Definition 2.16 (Calabi–Yau manifold II, [Lot22, Def. 3.6])**.** A **Calabi–Yau manifold** is a Riemannian manifold (M, g) of dimension $2n$ with $\mathrm{Hol}(g) \subset \mathrm{SU}(n)$.

Let (M^{2n}, g) be a Calabi–Yau manifold. At a point $p \in M$, we consider the nonzero $(n,0)$-form $\Omega_p \in \Lambda^{n,0}(T_p^*M)$ derived from $\widetilde{\Omega}$ on $\mathbb{C}^n$, which is invariant under $\mathrm{SU}(n)$. By [Lot22, Thm. 3.8], we have

$$|\widetilde{\Omega}(e_1, \ldots, e_n)| \leq 1$$

for any orthonormal vectors $e_1, \ldots, e_n \in \mathbb{C}^n$ with equality if and only if the vectors span a **Lagrangian** plane in $\mathbb{C}^n$, that is, $\widetilde{\omega}|_{\mathrm{span}\{e_1,\ldots,e_n\}} = 0$. In particular, we have $\widetilde{\Omega}(e_1, \ldots, e_n) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ whenever $\mathrm{span}\{e_1, \ldots, e_n\}$ is Lagrangian. This motivates the following observation: For every $\theta \in \mathbb{R}$, we have

$$\mathrm{Re}\big(e^{-i\theta}\widetilde{\Omega}(e_1, \ldots, e_n)\big) \leq |\widetilde{\Omega}(e_1, \ldots, e_n)| \leq 1$$

with equality if and only if $\mathrm{span}\{e_1, \ldots, e_n\}$ is Lagrangian and $\mathrm{Im}(e^{-i\theta}\widetilde{\Omega}(e_1, \ldots, e_n)) = 0$. A plane $\mathrm{span}\{e_1, \ldots, e_n\}$ satisfying these two conditions is called **special Lagrangian** with phase $e^{i\theta}$. Extending Ω_p parallelly to Ω guarantees that $d\Omega = 0$. Moreover, as g and ω are parallel as well, the condition $\mathrm{Re}(e^{-i\theta}\Omega)|_V \leq \mathrm{vol}_V$ is preserved and holds true for every oriented tangent n-plane V on M with equality if and only if $\omega|_V = 0$ and $\mathrm{Im}(e^{-i\theta}\Omega)|_V = 0$. Thus, we obtain the following result.

Theorem 2.17 ([Lot22, Sec. 3.2])**.** *On a Calabi–Yau manifold* $(M, g, J, \omega, \Omega)$*, we have natural calibrations given by* $\mathrm{Re}(e^{-i\theta}\Omega)$ *for* $\theta \in \mathbb{R}$*. An oriented real* n*-dimensional submanifold* N *of* M *is calibrated by* $\mathrm{Re}(e^{-i\theta}\Omega)$ *if and only if it is special Lagrangian with phase* $e^{i\theta}$*. That is,*

$$\omega|_N = 0 \ (\textbf{\textit{Lagrangian}}) \quad \text{and} \quad \mathrm{Im}(e^{-i\theta}\Omega)|_N = 0 \ (\textbf{\textit{special}}\ \textit{Lagrangian}). \tag{2.2}$$

2.3.3. The octonions and the groups G_2 and $\mathrm{Spin}(7)$

Before moving on to the other two examples, let us have a look at the normed division algebra of octonions $\mathbb{O} \cong \mathbb{R}^8$ and its purely imaginary subspace $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. This recap covers essential notions that we need for what follows. See [HL82, Sec. IV.1, IV.B], [SW17] and [Kar20] for more details.

The octonions $\mathbb{O}$ equipped with octonionic multiplication and the standard inner product $\langle u, v\rangle = \mathrm{Re}(u\bar{v}) = \mathrm{Re}(\bar{u}v)$ form a normed nonassociative alternative algebra. In other words, it fulfills the weaker conditions $u(uv) = u^2 v$ and $(vu)u = vu^2$ for $u, v \in \mathbb{O}$. All octonions $u, v, w \in \mathbb{O}$ additionally satisfy $\overline{uv} = \bar{v}\bar{u}$,

$$\langle uv, w\rangle = \langle v, \bar{u}w\rangle = \langle u, w\bar{v}\rangle \qquad \text{and} \qquad \langle uv, uw\rangle = \langle vu, wu\rangle = \langle v, w\rangle |u|^2. \tag{2.3}$$

Furthermore, we have the identities

$$u(\bar{v}w) + v(\bar{u}w) = 2\langle u, v\rangle w \qquad \text{and} \qquad (u\bar{v})w + (u\bar{w})v = 2\langle v, w\rangle u, \tag{2.4}$$

which in particular imply $u\bar{v} + v\bar{u} = 2\langle u, v\rangle$.

There exist three different kinds of alternating multilinear brackets on $\mathbb{O}$:

- commutator: $[u, v] = uv - vu,$
- associator: $[u, v, w] = (uv)w - u(vw),$
- coassociator: $\frac{1}{2}[u, v, w, y] = -\langle v, wy\rangle u + \langle w, yu\rangle v - \langle y, uv\rangle w + \langle u, vw\rangle y;$

as well as three different kinds of cross products:

- two-fold: $u \times v = -\frac{1}{2}(\bar{u}v - \bar{v}u) = -\mathrm{Im}(\bar{u}v),$
- three-fold: $2u \times v \times w = u(\bar{v}w) - w(\bar{v}u),$
- four-fold: $4u \times v \times w \times y = \bar{u}(v \times w \times y) - \bar{v}(w \times y \times u) + \bar{w}(y \times u \times v) - \bar{y}(u \times v \times w).$

In particular, the last two simplify to $u \times v \times w = u(\bar{v}w)$ and $u \times v \times w \times y = \bar{u}(v(\bar{w}y))$ whenever u, v, w, y are orthogonal. Moreover, we make the following observations: First of all, the three brackets and the two-fold cross product all restrict to maps on $\mathrm{Im}\,\mathbb{O}$, where the latter simply gives the standard cross product on $\mathrm{Im}\,\mathbb{O}$. Second, the cross products are indeed multilinear, alternating and satisfy $|u \times v| = |u \wedge v|$, $|u \times v \times w| = |u \wedge v \wedge w|$ and $|u \times v \times w \times y| = |u \wedge v \wedge w \wedge y|$. Additionally, the three- and four-fold cross products are orthogonal to their arguments and the same holds true for the two-fold cross product when restricted to $\mathrm{Im}\,\mathbb{O}$, which justifies the terminology.

We now define the associative 3-form $\widetilde{\varphi} \in \Lambda^3(\operatorname{Im}\mathbb{O})^* \cong \Lambda^3(\mathbb{R}^7)^*$ and the coassociative 4-form $\widetilde{\psi} \in \Lambda^4(\operatorname{Im}\mathbb{O})^* \cong \Lambda^4(\mathbb{R}^7)^*$ on $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ as

$$\widetilde{\varphi}(u,v,w) = \frac{1}{2}\langle [u,v], w\rangle = \langle u \times v, w\rangle = \langle uv, w\rangle, \tag{2.5}$$

$$\widetilde{\psi}(u,v,w,y) = -\frac{1}{2}\langle [u,v,w], y\rangle = \langle u, v \times w \times y\rangle, \tag{2.6}$$

for $u,v,w,y \in \operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. In fact, $\widetilde{\psi} = *\widetilde{\varphi} \in \Lambda^4(\mathbb{R}^7)^*$ is the Hodge dual of $\widetilde{\varphi} \in \Lambda^3(\mathbb{R}^7)^*$ with respect to the given inner product. Furthermore, this leads us to the Cayley 4-form $\widetilde{\Phi} = 1^* \wedge (\widetilde{\varphi} \circ \pi_{1^\perp}) + (\widetilde{\psi} \circ \pi_{1^\perp}) \in \Lambda^4\mathbb{O}^* \cong \Lambda^4(\mathbb{R}^8)^*$ on $\mathbb{O} \cong \mathbb{R}^8$, which can be written as

$$\widetilde{\Phi}(u,v,w,y) = \langle u, v \times w \times y\rangle = -\langle u \times v \times w, y\rangle \tag{2.7}$$

for $u,v,w,y \in \mathbb{O} \cong \mathbb{R}^8$, where $\pi_{1^\perp} : \mathbb{O} \to \operatorname{Im}\mathbb{O}$ stands for the projection onto $\operatorname{Im}\mathbb{O} = 1^\perp$. By definition, $\widetilde{\Phi}$ is self-dual:

$$*\widetilde{\Phi} = *\big(1^* \wedge (\widetilde{\varphi} \circ \pi_{1^\perp})\big) + *(\widetilde{\psi} \circ \pi_{1^\perp}) = (\widetilde{\psi} \circ \pi_{1^\perp}) + 1^* \wedge (\widetilde{\varphi} \circ \pi_{1^\perp}) = \widetilde{\Phi}.$$

Example 2.18. Let $e_0, e_1, \ldots, e_7$ stand for the standard orthonormal basis of $\mathbb{R}^8$ with dual basis $e^0, e^1, \ldots, e^7$. We identify these vectors with the standard basis of $\mathbb{O}$ given by $1, \mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{e}, \mathrm{ie}, \mathrm{je}, \mathrm{ke}$. Then it makes sense to view $\mathbb{R}^7 \subset \mathbb{R}^8$ as the space spanned by $e_1, \ldots, e_7$, which corresponds to $\operatorname{Im}\mathbb{O}$. By applying the octonion multiplication rules (see Appendix C), we find that $\widetilde{\varphi}(u,v,w) = \langle uv, w\rangle$, $u,v,w \in \mathbb{R}^7$, takes the form

$$\widetilde{\varphi} = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}, \tag{2.8}$$

where $e^{123} = e^1 \wedge e^2 \wedge e^3$ etc. On the other hand, we have

$$\widetilde{\psi}(u,v,w,y) = \langle u, v \times w \times y\rangle = \langle u, v(\bar{w}y)\rangle = -\langle u, v(wy)\rangle$$

whenever $u,v,w,y \in \mathbb{R}^7$ are orthogonal, which yields

$$\widetilde{\psi} = e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}.$$

Note that $\widetilde{\psi}$ is indeed the Hodge dual of $\widetilde{\varphi}$. From the above formulas, we get

$$\begin{aligned}\widetilde{\Phi} = e^0 \wedge \widetilde{\varphi} + \widetilde{\psi} = e^{0123} &+ e^{0145} - e^{0167} + e^{0246} + e^{0257} + e^{0347} - e^{0356} \\ &+ e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}.\end{aligned} \tag{2.9}$$

We define the groups $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ as the stabilizers of $\widetilde{\varphi}$ and $\widetilde{\Phi}$, respectively,

$$\mathrm{G}_2 = \{A \in \mathrm{GL}_7(\mathbb{R}) \mid A^*\widetilde{\varphi} = \widetilde{\varphi}\},$$
$$\mathrm{Spin}(7) = \{A \in \mathrm{GL}_8(\mathbb{R}) \mid A^*\widetilde{\Phi} = \widetilde{\Phi}\},$$

which leads us to the topic of calibrated submanifolds in manifolds of *exceptional holonomy*. (For more details on the classification of Riemannian holonomy groups, especially *Berger's list* [Ber55, Sec. 3, Thm. 3], see, for example, [Joy07, Ch. 3.4].)

2.3.4. G_2-manifolds and (co-)associative submanifolds

The first *exceptional* case refers to G_2*-manifolds*. Using the above characterization of the group G_2, it is evident by the holonomy principle that the following two definitions are equivalent.

Definition 2.19 (G_2-manifold I, II; [Kar20, Def. 4.10], [Joy07, Def. 11.1.2], [Lot22, Def. 4.3])**.** A G_2**-manifold** is a 7-dimensional Riemannian manifold (M, g) equipped with a parallel 3-form $\varphi \in \Omega^3(M)$ which can be identified with $\widetilde{\varphi}$ (2.5) at every point. Such a form φ is called a **parallel** or **torsion-free** G_2**-structure** on (M, g). We write (M, g, φ) for the given data. Equivalently, a G_2**-manifold** is a Riemannian manifold (M^7, g) with $\mathrm{Hol}(g) \subset \mathrm{G}_2$.

Example 2.20. The model example of a G_2-manifold is $\mathbb{R}^7$ equipped with the standard metric and the 3-form $\varphi \in \Omega^3(\mathbb{R}^7)$ defined by $\varphi|_p = \widetilde{\varphi} \in \Lambda^3(T_p^*\mathbb{R}^7) = \Lambda^3(\mathbb{R}^7)^*$ (2.8).

In fact, there exists a (not necessarily parallel) G_2-structure φ on (M, g) if and only if we can identify the tangent spaces T_pM, $p \in M$, with the purely imaginary octonions $\mathrm{Im}\,\mathbb{O}$ in a smoothly varying way. This follows from the fact that φ uniquely determines the metric g and the orientation vol_M via the relation

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = -6\, g(u, v)\, \mathrm{vol}_M, \qquad u, v \in T_pM,\ p \in M^7$$

where $\lrcorner$ stands for the interior product (see [Kar05, Sec. 2.3], although we differ by a sign here due to the opposite choice of orientation). Given this data, we naturally obtain a two-fold cross product on T_pM via $(u \times v)^\flat = v \lrcorner u \lrcorner \varphi$. This further leads to a product on the space $\mathbb{R} \oplus T_pM$, defined by

$$(u_1 + u)(v_1 + v) = u_1 v_1 - g(u, v) + u_1 v + v_1 u + u \times v \tag{2.10}$$

for $u_1 + u, v_1 + v \in \mathbb{R} \oplus T_pM$ (cf. [SW17, Thm. 5.4]). As in Subsubsection 2.3.3, we find three multilinear alternating brackets on T_pM, including the associator $[\cdot,\cdot,\cdot]$ and the coassociator $[\cdot,\cdot,\cdot,\cdot]$. These observations demonstrate that the existence of such a φ goes hand in hand with smoothly varying structures on the tangent spaces, that mirror those on the purely imaginary octonions.

Given a G_2-manifold (M, g, φ), we notice that the Hodge dual $\psi = *\varphi \in \Omega^4(M)$ of φ is also parallel and can be identified with $\widetilde{\psi}$ (2.6) at every point. As both φ and ψ are G_2-invariant, the following result is not surprising.

Theorem 2.21 ([HL82, Thm. 1.4, 1.16])**.** *Let (M, g, φ) be a G_2-manifold. Then $\varphi \in \Omega^3(M)$ and $\psi = *\varphi \in \Omega^4(M)$ are calibrations.*

Proof. As φ and ψ are parallel, they are in particular closed. Let $e_1, e_2, e_3 \in T_pM$ be orthonormal tangent vectors to M. Then φ satisfies

$$\varphi(e_1, e_2, e_3) = g(e_1 \times e_2, e_3) \le |e_1 \times e_2||e_3| = |e_1 \wedge e_2||e_3| = 1 \tag{2.11}$$

by the Cauchy–Schwarz inequality. Therefore, φ is a calibration, and by Lemma 2.4, so is $\psi = *\varphi$. ∎

We call the corresponding calibrated submanifolds **associative 3-folds** and **coassociative 4-folds**, respectively. There are different characterizations of such submanifolds and some of them are captured in the following two propositions.

Proposition 2.22 (Associative submanifolds; [HL82, Sec. IV.1], [KM05, Prop. 2.3])**.** *Let (M^7, g, φ) be a G_2-manifold with $\psi = *\varphi$ and $E^3 \subset M^7$ be an oriented submanifold. Then the following are equivalent (up to a change of orientation):*

(i) The submanifold E^3 is associative in M^7, that is, $\varphi|_E = \mathrm{vol}_E$.

(ii) The tangent space $TE \subset TM$ of E is preserved by the two-fold cross product.

(iii) The associator $[\cdot,\cdot,\cdot]$ vanishes on E.

(iv) At every point $p \in E$, we have $u \lrcorner v \lrcorner w \lrcorner \psi = 0$ for some basis $\{u, v, w\}$ of T_pE.

Proof. We start by proving the equivalence (i) ⇔ (ii). Let $p \in E$. By (2.11), the oriented tangent 3-plane T_pE spanned by orthonormal tangent vectors $e_1, e_2, e_3 \in T_pM$ is calibrated by φ if and only if we have $e_i \times e_j = e_k$ for all even permutations (i, j, k) of $(1, 2, 3)$. Due to the multilinearity and the alternating property of cross products, this condition implies that the two-fold cross product of any two vectors in T_pE lies again in T_pE. Conversely, suppose that the two-fold cross product preserves T_pE. As $e_i \times e_j$, $i \neq j$, has unit length and is orthogonal to both e_i and e_j, we must have $e_i \times e_j = \varepsilon_{ijk} e_k$, $\varepsilon_{ijk} \in \{\pm 1\}$, for all permutations (i, j, k) of $(1, 2, 3)$. Since φ is alternating, so is the sign ε_{ijk}. Thus, we can equip T_pE with an orientation such that $\varepsilon_{ijk} = +1$ for all even permutations. Since the two-fold cross product varies smoothly on M, we can pick a global orientation satisfying this condition. As a result, E becomes associative in M^7.

Before proving (i) ⇔ (iii), let us derive a general identity, which relates the associative 3-form to the associator. Let $p \in M$ and $u, v, w \in T_pM$ be orthogonal tangent vectors. Then

$$\begin{aligned}\varphi(u, v, w) &= \varphi(v, w, u) = g(vw, u) = \mathrm{Re}\big(\bar{u}(vw)\big) = -\mathrm{Re}\big(u(vw)\big) = \mathrm{Re}\big(u(\bar{v}w)\big) \\ &= \mathrm{Re}(u \times v \times w).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathrm{Im}(u \times v \times w) &= \mathrm{Im}\big(u(\bar{v}w)\big) = -\mathrm{Im}\big(u(vw)\big) \\ &= -\frac{1}{2}\big(u(vw) - \overline{u(vw)}\big) = -\frac{1}{2}\big(u(vw) - (\bar{w}\bar{v})\bar{u}\big) = -\frac{1}{2}\big(u(vw) + (wv)u\big) \\ &= -\frac{1}{2}\big(u(vw) - (uv)w\big) = \frac{1}{2}[u, v, w].\end{aligned}$$

Above, we used the identity $(uv)w = -(vu)w = (vw)u = -(wv)u$, obtained by applying (2.4) multiple times. As the three-fold cross product, φ and the associator are all alternating, the resulting identities

$$\mathrm{Re}(u \times v \times w) = \varphi(u, v, w) \qquad \text{and} \qquad \mathrm{Im}(u \times v \times w) = \frac{1}{2}[u, v, w]$$

even hold true for general $u, v, w \in T_pM$. Using that $|u \times v \times w| = |u \wedge v \wedge w|$, we get

$$\varphi(u, v, w)^2 + \frac{1}{4}\big|[u, v, w]\big|^2 = |u \wedge v \wedge w|^2. \tag{2.12}$$

Now let $p \in E$ and e_1, e_2, e_3 be an oriented orthonormal basis for T_pE. As $|e_1 \wedge e_2 \wedge e_3| = 1$, (2.12) implies that $\varphi(e_1, e_2, e_3) = \pm 1$ is equivalent to $[e_1, e_2, e_3] = 0$. After changing the orientation if necessary, this shows that T_pE is calibrated by φ if and only if the associator vanishes on T_pE. As φ varies smoothly on M, there exists a global orientation that satisfies this, which proves (i) $\Leftrightarrow$ (iii) (up to a change of orientation).

Lastly, we prove (iii) $\Leftrightarrow$ (iv). Let $p \in E$ and $u, v, w \in T_pM$ be a basis for T_pE. By (2.10), we have

$$\begin{aligned}
[u, v, w] &= (uv)w - u(vw) \\
&= \big(-g(u, v) + u \times v\big)w - u\big(-g(v, w) + v \times w\big) \\
&= -g(u, v)w + g(v, w)u + (u \times v)w - u(v \times w) \\
&= -g(u, v)w + g(v, w)u - g(u \times v, w) + g(u, v \times w) + (u \times v) \times w - u \times (v \times w) \\
&= -g(u, v)w + g(v, w)u - \varphi(u, v, w) + \varphi(v, w, u) + (u \times v) \times w - u \times (v \times w) \\
&= -g(u, v)w + g(v, w)u + (u \times v) \times w - u \times (v \times w),
\end{aligned}$$

where we used that φ is alternating. From [Kar05, Lemma 2.4.3], we know

$$u \times (v \times w) = -g(u, v)w + g(u, w)v - (u \lrcorner v \lrcorner w \lrcorner \psi)^\sharp.$$

From this, we also get

$$-(u \times v) \times w = w \times (u \times v) = -g(u, w)v + g(v, w)u - (u \lrcorner v \lrcorner w \lrcorner \psi)^\sharp.$$

Combining all of this yields

$$[u, v, w] = 2(u \lrcorner v \lrcorner w \lrcorner \psi)^\sharp.$$

As a result, T_pE is calibrated by φ if and only if we have $u \lrcorner v \lrcorner w \lrcorner \psi = 0$. As $p \in E$ was arbitrary, this completes the proof. □

Proposition 2.23 (Coassociative submanifolds, [HL82, Sec. IV.1])**.** *Let (M^7, g, φ) be a G_2-manifold with $\psi = *\varphi$ and $F^4 \subset M^7$ be an oriented submanifold. Then the following are equivalent (up to a change of orientation):*

(i) The submanifold F^4 is coassociative in M^7, that is, $\psi|_F = \mathrm{vol}_F$.

(ii) The two-fold cross product $u \times v$ is orthogonal to T_pF for all $u, v \in T_pF$, $p \in F$.

(iii) The coassociator $[\cdot, \cdot, \cdot, \cdot]$ vanishes on F.

(iv) $\varphi|_F = 0$.

Proof. Let us first derive a general identity, which relates the coassociative 4-form ψ to the coassociator and is key in the proof of (i) $\Leftrightarrow$ (iii). Subsequently, we prove the equivalences (iii) $\Leftrightarrow$ (iv) and (iv) $\Leftrightarrow$ (ii). Let $p \in M$ and $u, v, w, y \in T_pM$ be orthogonal tangent vectors. Then

$$\psi(u, v, w, y) = g(u, v \times w \times y) = g\big(u, v(\bar{w}y)\big) = \mathrm{Re}\big(\bar{u}(v(\bar{w}y))\big) = \mathrm{Re}(u \times v \times w \times y).$$

On the other hand, we have

$$\begin{aligned} g\big(\mathrm{Im}(u \times v \times w \times y), u\big) &= g(u \times v \times w \times y, u) = g\big(\bar{u}(v(\bar{w}y)), u\big) = -g\big(u(\bar{v}(wy)), u\big) \\ &= -g\big(\bar{v}(wy), 1\big)|u|^2 = -g(wy, v)|u|^2 \\ &= \frac{1}{2}\, g\big([u, v, w, y], u\big), \end{aligned}$$

where we used (2.3) in the second line. As the four-fold cross product and the coassociator are alternating, this proves that the components of $\mathrm{Im}(u \times v \times w \times y)$ and $\frac{1}{2}[u, v, w, y]$ in the directions of u, v, w and y are equal. In other words, the two terms differ at most by a tangent vector orthogonal to u, v, w and y. That is, $\mathrm{Im}(u \times v \times w \times y) = \frac{1}{2}[u, v, w, y] + a$ for some $a \in \mathrm{span}\{u, v, w, y\}^{\perp} \subset T_pM$. We will show that $a = 0$. By definition, $[u, v, w, y]$ lies in $\mathrm{span}\{u, v, w, y\}$ and is therefore orthogonal to a. Hence, we get

$$g\big(\mathrm{Im}(u \times v \times w \times y), a\big) = \frac{1}{2}\, g\big([u, v, w, y], a\big) + |a|^2 = |a|^2.$$

In other words, a is equal to zero if and only if $g(\mathrm{Im}(u \times v \times w \times y), a) = 0$. We have

$$g\big(\mathrm{Im}(u \times v \times w \times y), a\big) = g\big(\bar{u}(v(\bar{w}y)), a\big) = g\big(u(v(wy)), a\big).$$

Applying (2.4) six times gives $g(u(v(wy)), a) = g(y(w(vu)), a)$. On the other hand, we obtain

$$\begin{aligned} g\big(u(v(wy)), a\big) &= g\big(\bar{a}(u(v(wy))), 1\big) = -g\big(a(u(v(wy))), 1\big) = -g\big(u(v(w(ya))), 1\big) \\ &= -g\big(a, \bar{y}(\bar{w}(\bar{v}\bar{u}))\big) = -g\big(a, y(w(vu))\big) \end{aligned}$$

by using (2.3) and (2.4) multiple times. We deduce that $g(\mathrm{Im}(u \times v \times w \times y), a) = g(y(w(vu)), a) = -g(a, y(w(vu)))$, which proves that $\mathrm{Im}(u \times v \times w \times y)$ is orthogonal to a and therefore that $a = 0$. As the four-fold cross product, ψ and the coassociator are alternating, the resulting identities

$$\mathrm{Re}(u \times v \times w \times y) = \psi(u, v, w, y) \qquad \text{and} \qquad \mathrm{Im}(u \times v \times w \times y) = \frac{1}{2}[u, v, w, y]$$

even hold true for general $u, v, w, y \in T_pM$. Using that $|u \times v \times w \times y| = |u \wedge v \wedge w \wedge y|$, we get

$$\psi(u, v, w, y)^2 + \frac{1}{4}\big|[u, v, w, y]\big|^2 = |u \wedge v \wedge w \wedge y|^2. \tag{2.13}$$

Now let $p \in F$ and $e_1, \ldots, e_4$ be an oriented orthonormal basis for T_pF. Given that $|e_1 \wedge e_2 \wedge e_3 \wedge e_4| = 1$, (2.13) implies that $\psi(e_1, e_2, e_3, e_4) = \pm 1$ is equivalent to

$[e_1, e_2, e_3, e_4] = 0$. After changing the orientation if necessary, this proves that T_pF is calibrated by ψ if and only if the coassociator vanishes on T_pF. As ψ varies smoothly on M, there exists a global orientation that satisfies this, which proves (i) $\Leftrightarrow$ (iii) (up to a change of orientation).

Let us proceed to the proof of (iii) $\Leftrightarrow$ (iv). By definition, we have

$$\frac{1}{2}[u, v, w, y] = -\varphi(v, w, y)u + \varphi(w, y, u)v - \varphi(y, u, v)w + \varphi(u, v, w)y \tag{2.14}$$

for all $u, v, w, y \in T_pM$, $p \in M$. Now let $p \in F$ and u, v, w, y be a basis for T_pF. As the coassociator is alternating, it vanishes on T_pF if and only if $[u, v, w, y] = 0$, which is equivalent to $\varphi(v, w, y) = \varphi(w, y, u) = \varphi(y, u, v) = \varphi(u, v, w) = 0$ by (2.14). Since φ is alternating too, this condition is equivalent to $\varphi|_{T_pF} = 0$. As $p \in F$ was arbitrary, this proves (iii) $\Leftrightarrow$ (iv).

Lastly, the equivalence (iv) $\Leftrightarrow$ (ii) follows directly from the definition $\varphi(u, v, w) = g(u \times v, w)$ for $u, v, w \in T_pM$, $p \in M$. □

2.3.5. Spin(7)-manifolds and Cayley submanifolds

The second class of manifolds with *exceptional holonomy* comprises Spin(7)-*manifolds*. As in the G_2 case, the equivalence of the following two definitions is a direct consequence of the characterization of the group Spin(7) along with the holonomy principle.

Definition 2.24 (Spin(7)-manifold I, II; [Joy07, Def. 11.4.2], [Lot22, Def. 4.14])**.** A Spin(7)**-manifold** is an 8-dimensional Riemannian manifold (M, g) equipped with a parallel 4-form $\Phi \in \Omega^4(M)$ which can be identified with $\widetilde{\Phi}$ (2.7) at every point. Such a form Φ is called a **parallel** or **torsion-free** Spin(7)**-structure** on (M, g). We write (M, g, Φ) for the given data. Equivalently, a Spin(7)**-manifold** is a Riemannian manifold (M^8, g) with $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$.

Example 2.25. The model example of a Spin(7)-manifold is $\mathbb{R}^8$ equipped with the standard metric and the 4-form $\Phi \in \Omega^4(\mathbb{R}^8)$ defined by $\Phi|_p = \widetilde{\Phi} \in \Lambda^4(T_p^*\mathbb{R}^8) = \Lambda^4(\mathbb{R}^8)^*$ (2.9).

Similarly to the G_2 case, there exists a (not necessarily parallel) Spin(7)-structure Φ on (M, g) if and only if we can identify the tangent spaces T_pM, $p \in M$, with the octonions $\mathbb{O}$ in a smoothly varying way. This is due to the fact that Φ uniquely determines the metric g and the orientation vol_M via the relation

$$(u \lrcorner v_1 \lrcorner \Phi) \wedge (u \lrcorner v_2 \lrcorner \Phi) \wedge \Phi = -6\big(g(u, u)g(v_1, v_2) - g(u, v_1)g(u, v_2)\big)\,\mathrm{vol}_M \tag{2.15}$$

for $u, v_1, v_2 \in T_pM$, $p \in M^8$ (see [Kar05, Sec. 4.3], although the presented formula differs by a sign). Given this data, we naturally obtain a three-fold cross product X on T_pM via

$$X(u, v, w)^\flat = (-u \times v \times w)^\flat = w \lrcorner v \lrcorner u \lrcorner \Phi, \tag{2.16}$$

where we introduce a sign to match the convention used in [KM05]. This further leads to a product on T_pM defined by

$$uv = -X(u, e_0, v) + g(u, e_0)v + g(v, e_0)u - g(u, v)e_0 \tag{2.17}$$

for some fixed unit vector $e_0 \in T_pM$ (cf. [SW17, Thm. 5.20]). (This is simply a generalization of the product defined in (2.10), where $e_0 = 1 \in \mathbb{R} \subset \mathbb{R} \oplus T_pM^7$.) Following Subsubsection 2.3.3, we find three alternating multilinear brackets and two additional kinds of cross products on T_pM, including a four-fold cross product $\cdot \times \cdot \times \cdot \times \cdot$. These observations demonstrate that the existence of such a Φ goes hand in hand with smoothly varying structures on the tangent spaces, that mirror those on the octonions. As a result, we also obtain a natural splitting of the tangent vectors into real and imaginary parts.

As a parallel Spin(7)-structure Φ is in particular Spin(7)-invariant, the following result is what we would expect.

Theorem 2.26 ([HL82, Thm. 1.24]). *Let (M, g, Φ) be a Spin(7)-manifold. Then $\Phi \in \Omega^4(M)$ is a calibration.*

Proof. As Φ is parallel, it is in particular closed. Let $e_1, \dots, e_4 \in T_pM$ be orthonormal tangent vectors to M. Then Φ satisfies

$$\Phi(e_1, e_2, e_3, e_4) = g\big(X(e_1, e_2, e_3), e_4\big) \leq |X(e_1, e_2, e_3)||e_4| = |e_1 \wedge e_2 \wedge e_3||e_4| = 1 \quad (2.18)$$

by the Cauchy–Schwarz inequality. Therefore, Φ is a calibration. □

We refer to the corresponding calibrated submanifolds as **Cayley 4-folds**. The following proposition captures some equivalent characterizations of them.

Proposition 2.27 (Cayley submanifolds; [HL82, Sec. IV.1], [KM05, Prop. 2.5]). *Let (M^8, g, Φ) be a Spin(7)-manifold and $F^4 \subset M^8$ be an oriented submanifold. Then the following are equivalent (up to a change of orientation):*

(i) The submanifold F^4 is Cayley in M^8, that is, $\Phi|_F = \mathrm{vol}_F$.

(ii) The tangent space $TF \subset TM$ of F is preserved by the three-fold cross product X.

(iii) At every point $p \in F$, we have $\mathrm{Im}(u \times v \times w \times y) = 0$ for some basis $\{u, v, w, y\}$ of T_pF.

(iv) At every point $p \in F$, the rank 7 bundle valued 4-form η on M defined by

$$\begin{aligned}\eta(u, v, w, y) = {} & u^\flat \wedge X(v, w, y)^\flat + v^\flat \wedge X(w, u, y)^\flat + w^\flat \wedge X(u, v, y)^\flat + y^\flat \wedge X(v, u, w)^\flat \\ & + u \lrcorner X(v, w, y) \lrcorner \Phi + v \lrcorner X(w, u, y) \lrcorner \Phi + w \lrcorner X(u, v, y) \lrcorner \Phi + y \lrcorner X(v, u, w) \lrcorner \Phi\end{aligned}$$

vanishes for some basis $\{u, v, w, y\}$ of T_pF.

Proof. We start by proving the equivalence (i) ⇔ (ii). Let $p \in F$. By (2.18), the oriented tangent 4-plane T_pF spanned by orthonormal tangent vectors $e_1, \dots, e_4 \in T_pM$ is calibrated by Φ if and only if we have $X(e_i, e_j, e_k) = e_l$ for all even permutations (i, j, k, l) of $(1, 2, 3, 4)$. Due to the multilinearity and the alternating property of cross products, this condition implies that the three-fold cross product of any three vectors in T_pF lies again in T_pF. Conversely, suppose that the three-fold cross product preserves T_pF. As $X(e_i, e_j, e_k)$ has unit length and is orthogonal to e_i, e_j and e_k whenever i, j, k are distinct, we must have $X(e_i, e_j, e_k) = \varepsilon_{ijkl} e_l$, $\varepsilon_{ijkl} \in \{\pm 1\}$, for all permutations (i, j, k, l) of $(1, 2, 3, 4)$. Since Φ is alternating, so is the sign ε_{ijkl}. Thus, we can equip T_pF with an orientation such that $\varepsilon_{ijkl} = +1$ for all even permutations. Since the three-fold cross product varies smoothly on M, we can pick a global orientation satisfying this condition. As a result, F becomes Cayley in M^8.

We move on to the proof of (i) $\Leftrightarrow$ (iii). Let $p \in M$ and $u, v, w, y \in T_pM$ be orthogonal tangent vectors. Similarly to the coassociative case, we compute

$$\Phi(u, v, w, y) = g(u, v \times w \times y) = g\big(u, v(\bar{w}y)\big) = \mathrm{Re}\big(\bar{u}(v(\bar{w}y))\big) = \mathrm{Re}(u \times v \times w \times y).$$

As the four-fold cross product and Φ are both alternating, the identity $\Phi(u, v, w, y) = \mathrm{Re}(u \times v \times w \times y)$ even holds true for general $u, v, w, y \in T_pM$. Using that $|u \times v \times w \times y| = |u \wedge v \wedge w \wedge y|$, we get

$$\Phi(u, v, w, y)^2 + |\mathrm{Im}(u \times v \times w \times y)|^2 = |u \wedge v \wedge w \wedge y|^2 \tag{2.19}$$

for all $u, v, w, y \in T_pM$, $p \in M$. Now let $p \in F$ and $e_1, \ldots, e_4$ be an oriented orthonormal basis for T_pF. As $|e_1 \wedge e_2 \wedge e_3 \wedge e_4| = 1$, (2.19) implies that $\Phi(e_1, e_2, e_3, e_4) = \pm 1$ is equivalent to $\mathrm{Im}(e_1 \times e_2 \times e_3 \times e_4) = 0$. After changing the orientation if necessary, this shows that T_pF is calibrated by Φ if and only if the purely imaginary four-fold cross product vanishes on T_pF. As Φ varies smoothly on M, there exists a global orientation that satisfies this, which proves (i) $\Leftrightarrow$ (iii) (up to a change of orientation).

Lastly, we prove (iii) $\Leftrightarrow$ (iv). Let $e_0, \ldots, e_7$ be an orthonormal basis for T_pM, $p \in M$, where e_0 represents the multiplicative identity spanning $\mathrm{Re}(T_pM)$. Define a 4-form $\widetilde{\eta}$ with values in TM by

$$\begin{aligned}\widetilde{\eta}(u, v, w, y) &= -4\,\mathrm{Im}(u \times v \times w \times y) \\ &= \mathrm{Im}\big(\bar{u}X(v, w, y) - \bar{v}X(w, y, u) + \bar{w}X(y, u, v) - \bar{y}X(u, v, w)\big) \\ &= \mathrm{Im}\big(\bar{u}X(v, w, y) + \bar{v}X(w, u, y) + \bar{w}X(u, v, y) + \bar{y}X(v, u, w)\big)\end{aligned}$$

for $u, v, w, y \in T_pM$, $p \in M$. By (2.17), we have

$$\begin{aligned}\mathrm{Im}(\bar{u}v) &= \mathrm{Im}\big(-X(\bar{u}, e_0, v) + g(\bar{u}, e_0)v + g(v, e_0)\bar{u} - g(\bar{u}, v)e_0\big) \\ &= X(u, e_0, v) + g(u, e_0)\,\mathrm{Im}(v) - g(v, e_0)\,\mathrm{Im}(u) \\ &= \sum_{k=1}^{7} \big(g\big(X(u, e_0, v), e_k\big) + g(u, e_0)g(v, e_k) - g(v, e_0)g(u, e_k)\big)e_k \\ &= \sum_{k=1}^{7} \big(\Phi(v, u, e_0, e_k) + u^\flat(e_0)v^\flat(e_k) - v^\flat(e_0)u^\flat(e_k)\big)e_k \\ &= \sum_{k=1}^{7} \big((u \lrcorner v \lrcorner \Phi + u^\flat \wedge v^\flat)(e_0, e_k)\big)e_k,\end{aligned}$$

where we used that the three-fold cross product is orthogonal to its arguments and alternating to obtain

$$\mathrm{Im}\big(X(\bar{u}, e_0, v)\big) = X(\bar{u}, e_0, v) = X\big(\mathrm{Im}(\bar{u}), e_0, v\big) = -X\big(\mathrm{Im}(u), e_0, v\big) = -X(u, e_0, v).$$

This allows us to write $\widetilde{\eta}$ as

$$\widetilde{\eta}(u,v,w,y)=\sum_{k=1}^{7}\big(\eta(u,v,w,y)(e_0,e_k)\big)e_k \tag{2.20}$$

with

$$\begin{aligned}\eta(u,v,w,y)&=u^\flat\wedge X(v,w,y)^\flat+v^\flat\wedge X(w,u,y)^\flat+w^\flat\wedge X(u,v,y)^\flat+y^\flat\wedge X(v,u,w)^\flat\\&\quad+u\lrcorner X(v,w,y)\lrcorner\Phi+v\lrcorner X(w,u,y)\lrcorner\Phi+w\lrcorner X(u,v,y)\lrcorner\Phi+y\lrcorner X(v,u,w)\lrcorner\Phi.\end{aligned}$$

The latter is a 4-form on M with values in $\Lambda^2(T^*M)$. More precisely, it only takes values in a rank 7 subbundle and we will now outline how to see this. The bundle $\Lambda^2(T^*M)=\Lambda^2_7\oplus\Lambda^2_{21}$ of rank 28 splits into the subbundles

$$\Lambda^2_7=\big\{\alpha\in\Lambda^2(T^*M)\;\big|\;*(\Phi\wedge\alpha)=-3\alpha\big\}\quad\text{and}\quad\Lambda^2_{21}=\big\{\alpha\in\Lambda^2(T^*M)\;\big|\;*(\Phi\wedge\alpha)=\alpha\big\}$$

of rank 7 and 21, respectively (see [Kar05, Sec. 4.2], although the descriptions vary by a sign due to the opposite choice of orientation). Let $u,v\in T_pM$, $p\in M$. Using [Kar05, Lemma A.1] along with the fact that Φ is self-dual, we compute

$$u\lrcorner v\lrcorner\Phi=*\big(u^\flat\wedge*(v\lrcorner\Phi)\big)=-*(u^\flat\wedge v^\flat\wedge*\Phi)=-*(\Phi\wedge u^\flat\wedge v^\flat). \tag{2.21}$$

On the other hand, we have

$$*\big(\Phi\wedge(u\lrcorner v\lrcorner\Phi)\big)=-3u^\flat\wedge v^\flat-2u\lrcorner v\lrcorner\Phi. \tag{2.22}$$

To see this, we use the local form of Φ (2.9) to compute $|\Phi\wedge(e_i\lrcorner e_j\lrcorner\Phi)|^2=21$ for $i\neq j$. Furthermore, (2.15) and (2.21) imply

$$\begin{aligned}\Phi\wedge(e_i\lrcorner e_j\lrcorner\Phi)\wedge(e_i\lrcorner e_j\lrcorner\Phi)&=-6\,|e^i\wedge e^j|^2\operatorname{vol}_M=-6\operatorname{vol}_M,\\ \Phi\wedge(e_i\lrcorner e_j\lrcorner\Phi)\wedge e^i\wedge e^j&=-|e_i\lrcorner e_j\lrcorner\Phi|^2\operatorname{vol}_M=-3\operatorname{vol}_M.\end{aligned}$$

By multilinearity, we obtain (2.22). Combining this with (2.21), we get

$$*\big(\Phi\wedge(u^\flat\wedge v^\flat+u\lrcorner v\lrcorner\Phi)\big)=-3(u^\flat\wedge v^\flat+u\lrcorner v\lrcorner\Phi),$$

which proves that η only takes values in the rank 7 bundle Λ^2_7.

Now let $f_1,\dots,f_4$ be an orthogonal basis for T_pF at $p\in F$. Whenever $\eta(f_1,f_2,f_3,f_4)=0$, we also have $\widetilde{\eta}(f_1,f_2,f_3,f_4)=0$ by (2.20). Conversely, the equivalence (iii) $\Leftrightarrow$ (ii) implies that $\widetilde{\eta}(f_1,f_2,f_3,f_4)$ vanishes if and only if the three-fold cross product X of any three basis vectors lies in the span of the fourth one. As the wedge product on 1-forms and Φ are alternating, this forces every term in $\eta(f_1,f_2,f_3,f_4)$ to vanish. As a result, the condition $\eta(f_1,f_2,f_3,f_4)=0$ is equivalent to $\widetilde{\eta}(f_1,f_2,f_3,f_4)=0$. Since both η and $\widetilde{\eta}$ are multilinear, this also holds true for general bases u,v,w,y of T_pF, which completes the proof of (iii) $\Leftrightarrow$ (iv).

□

2.4. Classification of calibrations

Having discussed the four main examples of calibrated geometries, the question arises why these particular cases are of special importance and whether there are other noteworthy examples that we have not mentioned yet. In other words, we would like to classify calibrations. As we saw above, for every parallel calibration φ on Euclidean space $(\mathbb{R}^n, g)$ which is invariant under some group $G \subset \mathrm{O}(n)$, there exists a corresponding parallel calibration on any Riemannian n-manifold with holonomy in G. Therefore, classifying calibrations with constant coefficients on $\mathbb{R}^n$ also leads to a classification of parallel calibrations on manifolds of special holonomy. Fortunately, there have been significant efforts to classify constant calibrations on $\mathbb{R}^n$, and the most important results are compiled, for example, in [Joy07, Ch. 4.3]. Here, we provide a brief overview.

Let $\varphi \in \Lambda^k(\mathbb{R}^n)^*$ be a k-calibration and denote the Grassmannian of oriented k-planes in $\mathbb{R}^n$ by $\mathrm{Gr}_+(k, \mathbb{R}^n)$. Then every $V \in \mathrm{Gr}_+(k, \mathbb{R}^n)$ satisfies $\varphi|_V \leq \mathrm{vol}_V$, with equality whenever V is calibrated by φ. We denote the subset of calibrated planes by $\mathcal{F}_\varphi \subset \mathrm{Gr}_+(k, \mathbb{R}^n)$ and call it a **face** of $\mathrm{Gr}_+(k, \mathbb{R}^n)$. Moreover, we consider two k-calibrations $\varphi, \psi \in \Lambda^k(\mathbb{R}^n)^*$ equivalent whenever they are conjugate under $\mathrm{O}(n)$ or satisfy $\mathcal{F}_\varphi = \mathcal{F}_\psi$. Hence, our goal is to determine all possible nonempty faces $\mathcal{F}_\varphi$ arising from calibrations $\varphi \in \Lambda^k(\mathbb{R}^n)^*$, up to the action of $\mathrm{O}(n)$ on $\mathrm{Gr}_+(k, \mathbb{R}^n)$. To this end, it is important to note that the Hodge star operator satisfies $*\mathrm{Gr}_+(k, \mathbb{R}^n) = \mathrm{Gr}_+(n-k, \mathbb{R}^n)$, and $*\mathcal{F}_\varphi = \mathcal{F}_{*\varphi}$ by Lemma 2.4. Consequently, a classification of the faces of $\mathrm{Gr}_+(k, \mathbb{R}^n)$ also yields a classification of the faces of $\mathrm{Gr}_+(n-k, \mathbb{R}^n)$.

To begin with, the case $k = 1$ is trivial because $\mathrm{Gr}_+(1, \mathbb{R}^n)$ can be naturally identified with the unit sphere $S^{n-1} \subset \mathbb{R}^n$, where nonempty faces are simply single points. Harvey and Lawson [HL82, Thm. II.7.16] then classified the case $k = 2$. Combining this with our observations above, Joyce [Joy07, Thm. 4.3.2] provides a complete description of calibrations of degree $1, 2, n-2$ and $n-1$, which, in particular, classifies all calibrations on $\mathbb{R}^n$ for $n \leq 5$. Additionally, Dadok and Harvey [DH83] and Morgan [Mor85, Sec. 4] examined the case $(n, k) = (6, 3)$, and Harvey and Morgan [HM86, Thm. 6.2] the case $(n, k) = (7, 3)$. Thanks to their efforts, all constant calibrations on $\mathbb{R}^n$ for $n \leq 7$ are classified. Based on these results, Joyce [Joy07, Ch. 4.3] drew the following insightful conclusion: For all constant calibrations φ on $\mathbb{R}^n$ with $\dim \mathcal{F}_\varphi > 0$ for $n \leq 6$ and $\dim \mathcal{F}_\varphi > 3$ for $n = 7$, the submanifolds calibrated by φ are derived from one of the following: (1) complex curves in $\mathbb{C}^2$ or $\mathbb{C}^3$, (2) complex surfaces in $\mathbb{C}^3$, (3) special Lagrangian 3-folds in $\mathbb{C}^3$, (4) associative 3-folds in $\mathbb{R}^7$, or (5) coassociative 4-folds in $\mathbb{R}^7$. In other words, there are no additional interesting calibrated geometries in dimension $n \leq 7$ that we have missed.

For any $n \geq 8$, no complete classification of faces of $\mathrm{Gr}_+(k, \mathbb{R}^n)$ exists. Nevertheless, there are some interesting examples in $\Lambda^4_+(\mathbb{R}^8)^*$, such as Cayley 4-folds, special Lagrangian 4-folds, complex surfaces, complex Lagrangian surfaces and affine quaternionic lines [DHM88], [Joy07, Ch. 4.3]. In fact, these are all examples of Cayley 4-folds. Even associative 3-folds and coassociative 4-folds can be regarded as special cases: For A^3 associative and C^4 coassociative in $\mathbb{R}^7$, both $\mathbb{R} \times A^3$ and $\{c\} \times C^4$ for $c \in \mathbb{R}$ are Cayley in $\mathbb{R}^8 = \mathbb{R} \times \mathbb{R}^7$. Conversely, consider a Cayley 4-fold L^4 in $\mathbb{R}^8 = \mathrm{span}\{1\} \times \mathbb{R}^7$. If $\pi_{1^\perp} L^4 = L^4$, then L^4 is coassociative in $\mathbb{R}^7$. Otherwise, L^4 can be written as $\mathbb{R} \times A^3$ for some associative 3-fold A^3 in $\mathbb{R}^7$. This follows from the close relationship between the standard (co-)associative and Cayley forms, characterized by $\widetilde{\Phi} = 1^* \wedge (\widetilde{\varphi} \circ \pi_{1^\perp}) + (\widetilde{\psi} \circ \pi_{1^\perp})$ on $\mathbb{R}^8$.

In summary, special Lagrangian submanifolds, (co-)associative submanifolds and Cayley 4-folds are indeed particularly important for $n \leq 8$. For $n \leq 7$, they even stand out as the most interesting examples. Therefore, following the lead of [IKM05; KL12; KM05], we focus exclusively on these four kinds of calibrated geometries hereafter.

3. Review of previous constructions

In this section, we establish our setup and notation, while outlining the constructions of calibrated submanifolds by Ionel–Karigiannis–Min-Oo [IKM05], Karigiannis–Leung [KL12] and Karigiannis–Min-Oo [KM05].

3.1. The second fundamental form

Let $(X^n, g = \langle\cdot,\cdot\rangle)$ be a real n-dimensional Riemannian manifold with local coordinates $x = (x_1, \ldots, x_n)$. We consider some oriented immersed submanifold L^q with local coordinates $u = (u_1, \ldots, u_q)$ and immersion $L^q \subset X^n$ given by $x_i = x_i(u)$, $i = 1, \ldots, n$. We write $(\,)^T$ and $(\,)^N$ for the orthogonal projections to the tangent bundle TL and normal bundle NL of $L \subset X$, respectively. Throughout this thesis, ∇ always denotes the Levi-Civita connection on the tangent bundle TX of the ambient manifold X^n, unless stated otherwise. Let us now fix a point $u^* \in L$ and let $x^* = x(u^*) \in X$. By parallel transporting orthonormal bases of $T_{x^*}L$ and $N_{x^*}L$ via the induced connections on TL and NL, respectively, we obtain a local orthonormal frame $e_1, \ldots, e_q, \nu_{q+1}, \ldots, \nu_n$ for TX that satisfies

$$(\nabla_{e_i} e_j)|_{x^*}^T = 0 \qquad \text{and} \qquad (\nabla_{e_i} \nu_j)|_{x^*}^N = 0 \tag{3.1}$$

[IKM05, Sec. 2]. We refer to a frame fulfilling (3.1) as **normal coordinates** and always work with them unless we say otherwise.

Let the **second fundamental form** A of the immersion $L^q \subset X^n$ be defined as the bilinear operator

$$A : \Gamma(NL) \times \Gamma(TL) \to \Gamma(TL), \quad (\nu, w) \mapsto A^\nu(w) = (\nabla_w \nu)^T.$$

It is easy to check that for any normal vector field ν, A^ν is a symmetric linear operator and, hence, diagonalizable [IKM05, Sec. 2]. We use the following abbreviations:

$$A^\nu_{ij} = \langle A^\nu(e_i), e_j\rangle = A^\nu_{ji} \qquad \text{and} \qquad A^k_{ij} = A^{\nu_k}_{ij}.$$

Remark 3.1. There are different ways to define the second fundamental form. Above, we stated the definition used in [HL82] and [IKM05], which relates to the more common definition

$$A : \Gamma(TL) \times \Gamma(TL) \to \Gamma(NL), \quad (v, w) \mapsto A(v, w) = (\nabla_w v)^N$$

(see, e.g., [Wen22, Def. 28.2]) via

$$\begin{aligned}\langle A(v, w), \nu\rangle &= \langle (\nabla_w v)^N, \nu\rangle = \langle \nabla_w v, \nu\rangle = -\langle v, \nabla_w \nu\rangle = -\langle v, (\nabla_w \nu)^T\rangle \\ &= -\langle v, A^\nu(w)\rangle\end{aligned}$$

for every $\nu \in \Gamma(NL)$. Here we used that v is tangential, ν is normal, and that ∇ is the Levi-Civita connection. In fact, $-A^\nu$ is called the **Weingarten map** associated to ν [Wen22, Sec. 28.1] and uniquely determines A. Nevertheless, we stick to the sign conventions and terms used in [IKM05] and continue to refer to A^ν as the second fundamental form in the direction of ν, as the sign does not affect the results.

Remark 3.2. One can show that L is minimal if and only if its **mean curvature vector** H, defined by $H = \operatorname{Tr} A$, vanishes [Lot22, Def. 2.3]. This is equivalent to demanding $\operatorname{Tr} A^k = 0$ for all $k = q+1, \dots, n$ in our notation.

Finally, using normal coordinates (3.1) and the second fundamental form A, we obtain the identities

$$\nabla_{e_i} e^j = - \sum_{k=q+1}^{n} A^k_{ij} \nu^k \qquad \text{and} \qquad \nabla_{e_i} \nu^j = \sum_{l=1}^{q} A^j_{il} e^l \tag{3.2}$$

at x^*, where $e^1, \dots, e^q$ and $\nu^{q+1}, \dots, \nu^n$ are the dual coframes [IKM05, (2.3)].

3.2. Special Lagrangians in $T^*\mathbb{R}^n$ and T^*S^n

We begin by examining special Lagrangians in the cotangent space T^*X for X^n being either Euclidean space $\mathbb{R}^n$ or the n-dimensional sphere S^n. All theorems presented in this subsection were directly proved by verifying the two conditions in (2.2).

Before reviewing the constructions, we need to recall the definition of the *elementary symmetric polynomials* and introduce *austere submanifolds*. Consider a matrix $B \in \mathbb{R}^{q \times q}$ and let $I \in \mathbb{R}^{q \times q}$ denote the identity matrix. The **elementary symmetric polynomials** $\sigma_j(B)$ of B are defined by

$$\det(I + tB) = \sum_{j=0}^{q} t^j \sigma_j(B)$$

or, more explicitly, as

$$\sigma_j(B) = \sum_{1 \le i_1 < \dots < i_j \le n} \lambda_{i_1} \cdots \lambda_{i_j}$$

for $\lambda_1, \dots, \lambda_q \in \mathbb{C}$ representing the eigenvalues of B (cf. [KL12, (2.4)], [Bos13, Ch. 4.4]). In particular, we have $\sigma_0(B) = 1$, $\sigma_1(B) = \operatorname{Tr} B$ and $\sigma_q(B) = \det B$. An oriented immersed submanifold $L^q \subset X^n$ is called **austere** if all odd degree elementary symmetric polynomials of the second fundamental form A^ν of $L^q \subset X^n$ vanish for all $\nu \in \Gamma(NL)$ [HL82, Def. 3.15]. That is, $\sigma_{2j-1}(A^\nu) = 0$ for all $j = 1, \dots, \lceil q/2 \rceil$ and $\nu \in \Gamma(NL)$. By Remark 3.2, this condition is equivalent to L^q being minimal for $q = 1, 2$, but much stronger for $q \ge 3$.

Laying the foundation for many subsequent works, Harvey and Lawson utilized bundles to construct a versatile example of special Lagrangians. More explicitly, they viewed the total space of the cotangent bundle $T^*\mathbb{R}^n$ as $\mathbb{C}^n$ and equipped it with the canonical Kähler form $\widetilde{\omega}$ and the standard $(n,0)$-volume form $\widetilde{\Omega}$. This led them to the following result for any oriented immersed submanifold $L^q \subset \mathbb{R}^n$.

Theorem 3.3 ([HL82, Thm. III.3.11])**.** *The conormal bundle N^*L is special Lagrangian in $T^*\mathbb{R}^n$ with phase $e^{i\theta} = \pm i^{n-q}$ if and only if L^q is austere in $\mathbb{R}^n$.*

This construction was later generalized by "twisting" N^*L by a special section $\mu \in \Omega^1(L)$. More precisely, we define the space

$$X_\mu = \big\{ (x, \xi + \mu_x) \in T^*\mathbb{R}^n|_L \,\big|\, x \in L,\, \xi \in N^*_x L \big\},$$

which we sometimes mnemonically refer to it as "$N^*L + \mu$". This is a "twisting" of the conormal bundle N^*L over L obtained by affinely translating each fiber N^*_xL by a cotangent vector $\mu_x \in T^*_xL$. As X_μ is a smooth n-dimensional submanifold of $T^*\mathbb{R}^n \cong \mathbb{C}^n$, it is natural to wonder whether it is special Lagrangian. This question was first investigated by Borisenko [Bor93] for exact 1-forms $\mu = d\rho$, $\rho \in C^\infty(L)$, in the cases $(q, n) = (2, 3)$ and $(3, 4)$. Later, Karigiannis and Leung considered general q, n and μ. They obtained the following results.

Lemma 3.4 ([KL12, Prop. 1]). *The submanifold $N^*L + \mu$ is Lagrangian in $T^*\mathbb{R}^n$ if and only if $d\mu = 0$.*

Theorem 3.5 ([KL12, Thm. 1]). *Suppose $d\mu = 0$ and define $\phi = \frac{\pi}{2}(n - q) - \theta$. Let $B = (B_{ij})_{i,j=1,\dots,n}$ denote the symmetrized covariant derivative of μ, that is,*

$$B_{ij} = (\nabla_{e_i}\mu)(e_j) = \frac{1}{2}\big((\nabla_{e_i}\mu)(e_j) + (\nabla_{e_j}\mu)(e_i)\big).$$

Then $N^*L + \mu$ is special Lagrangian in $T^*\mathbb{R}^n$ with phase $e^{i\theta}$ if and only if

$$\operatorname{Im}\big(e^{i\phi}\det(I + iB)\big) = 0 \quad \text{and} \quad \sigma_j\big(A^\nu(I + iB)^{-1}\big) = (-1)^j\sigma_j\big(A^\nu(I - iB)^{-1}\big)$$

for all $j = 1, \dots, q$ and all normal vector fields $\nu \in \Gamma(NL)$.

In particular, this implies the following hands-on result for the case $q = 2$.

Corollary 3.6 ([KL12, Cor. 1]). *Suppose $q = 2$. Then $N^*L + \mu$ is special Lagrangian in $T^*\mathbb{R}^n$ with phase $e^{i\theta} = \pm i^{n-2}$ if and only if L^2 is minimal in $\mathbb{R}^n$ and $\mu \in \Omega^1(L)$ is harmonic, i.e., $\Delta_L\mu = dd^*\mu + d^*d\mu = 0$.*

Given the above constructions, the question arises whether they work similarly for other manifolds than $\mathbb{R}^n$, whose cotangent spaces still possess metrics of holonomy in $\mathrm{SU}(n)$. Indeed, Karigiannis and Min-Oo [KM05] derived an analogous result to Theorem 3.3 for the standard round sphere S^n. In order to do so, they first endowed the cotangent space T^*S^n with a Calabi–Yau structure following [Sző91; Ste93; Anc07]: We identify T^*S^n with the complex quadric

$$Q = \left\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{k=0}^{n} z_k^2 = 1\right\}$$

via the diffeomorphism

$$\Psi : T^*S^n \to Q, \quad (x, \xi) \mapsto x\cosh|\xi| + i\frac{\xi}{|\xi|}\sinh|\xi|, \tag{3.3}$$

which is equivariant with respect to $\mathrm{SO}_{n+1}(\mathbb{R}) \subset \mathrm{O}_{n+1}(\mathbb{C})$. (Above, the term $\frac{\xi}{|\xi|}\sinh|\xi|$ is to be interpreted as 0 when $|\xi| = 0$.) From this, T^*S^n inherits a natural complex structure. Moreover, utilizing the radial vector field

$$Z = z_0\frac{\partial}{\partial z_0} + \cdots + z_n\frac{\partial}{\partial z_n}$$

on $\mathbb{C}^{n+1}$, we obtain a nowhere vanishing holomorphic $(n,0)$-form by defining

$$\Omega = Z \lrcorner \operatorname{vol}_{\mathbb{C}^{n+1}} = Z \lrcorner (dz_0 \wedge \cdots \wedge dz_n).$$

Finally, we equip T^*S^n (thought of as Q) with the complete Ricci-flat Kähler metric derived in [Ste93], called Stenzel metric. By [Anc07, Lemma 2.1], the corresponding Kähler form is given by

$$\omega_{\mathrm{St}} = \frac{i}{2} \sum_{j,k=1}^{n} a_{jk} dz_j \wedge d\bar{z}_k$$

with

$$a_{jk} = \left(\delta_{jk} + \frac{z_j \bar{z}_k}{|z_0|^2}\right) v' + 2 \operatorname{Re}\left(\bar{z}_j z_k - \frac{\bar{z}_0}{z_0} z_j z_k\right) v'' \tag{3.4}$$

in a neighborhood of a point where $z_0 \neq 0$. Above, v is a function of $r = |z|$, defined by a certain differential equation which ensures that the metric is Ricci-flat. For our purposes, it suffices to know that $v'(r), v''(r) > 0$ for $r > 0$ [Ste93, Prop. 6]. Based on these arrangements, we obtain a Calabi–Yau structure on T^*S^n, which justifies studying special Lagrangians and stating the following theorem.

Theorem 3.7 ([KM05, Thm. 3.1])**.** *Let $L^q \subset S^n$ be an oriented immersed submanifold. Then N^*L is special Lagrangian in T^*S^n with phase $e^{i\theta} = \pm i^{n-q}$ if and only if L^q is austere in S^n.*

The question that remains open and that we will address in Subsection 4.1 is the following.

Question 3.8. Let $L^q \subset S^n$ be an oriented immersed submanifold and $\mu \in \Omega^1(L)$. Consider

$$X_\mu = \left\{(x, \xi + \mu_x) \in T^*S^n|_L \,\middle|\, x \in L,\, \xi \in N_x^*L\right\} \qquad (\text{“}N^*L + \mu\text{”}).$$

Under what conditions on L and μ is $N^*L + \mu$ special Lagrangian in T^*S^n?

3.3. (Co-)associative submanifolds of $\Lambda^2_-(T^*X)$ for $X^4 = \mathbb{R}^4, S^4, \mathbb{CP}^2$

Let us proceed to associative and coassociative submanifolds of the space of anti-self-dual 2-forms $\Lambda^2_-(T^*X)$, where X^4 represents either Euclidean space $\mathbb{R}^4$, the 4-dimensional sphere S^4 or the complex projective plane $\mathbb{CP}^2$. To begin with, we revisit the bundle constructions in [IKM05] leading to candidates E and F for (co-)associative submanifolds of $\Lambda^2_-(T^*\mathbb{R}^4)$. In this case, the ambient manifold is naturally isomorphic to $\mathbb{R}^7$ and possesses a canonical parallel G_2-structure φ [BS89].

Consider an oriented immersed submanifold $L^2 \subset \mathbb{R}^4$ and fix some oriented local orthonormal adapted frame (e_1, e_2, ν_3, ν_4) along L with dual coframe (e^1, e^2, ν^3, ν^4). The restricted bundle $\Lambda^2_-(T^*\mathbb{R}^4)|_L$ is locally trivialized by the sections $f^1 = e^1 \wedge e^2 - \nu^3 \wedge \nu^4$, $f^2 = e^1 \wedge \nu^3 - \nu^4 \wedge e^2$ and $f^3 = e^1 \wedge \nu^4 - e^2 \wedge \nu^3$. Since f^1 is invariant under change of coordinates, it is globally defined on L. In fact, it can be written as $f^1 = \operatorname{vol}_L - *_{\mathbb{R}^4} \operatorname{vol}_L$. It thus spans a rank 1 bundle $E = \operatorname{span}\{f^1\}$, whose orthogonal complement $F = E^\perp$ in $\Lambda^2_-(T^*\mathbb{R}^4)|_L$ is locally represented by $F \overset{\text{loc}}{=} \operatorname{span}\{f^2, f^3\}$. The total spaces of these bundles are 3- and 4-dimensional submanifolds of $\Lambda^2_-(T^*\mathbb{R}^4)$, respectively. Ionel, Karigiannis and Min-Oo [IKM05] derived necessary and sufficient conditions on L^2 for E (F) to be associative (coassociative) in $\mathbb{R}^7$. As their results are a special case of the "twisted" version in the subsequent paper [KL12], we go straight to the constructions therein.

Karigiannis and Leung examined the spaces

$$X^E_\sigma = \{(x, \eta + \sigma_x) \in \Lambda^2_-(T^*\mathbb{R}^4)|_L \,|\, x \in L,\, \eta \in E_x\} \qquad (\text{“}E + \sigma\text{”}),$$
$$X^F_\eta = \{(x, \eta_x + \sigma) \in \Lambda^2_-(T^*\mathbb{R}^4)|_L \,|\, x \in L,\, \sigma \in F_x\} \qquad (\text{“}\eta + F\text{”})$$

for sections $\sigma \in \Gamma(F)$ and $\eta \in \Gamma(E)$. This is a "twisting" of the bundle E (F) over L obtained by affinely translating each fiber E_x (F_x) by a vector $\sigma_x \in F_x$ ($\eta_x \in E_x$) in the orthogonal complement. The spaces TL and NL can be endowed with the natural complex structures locally given by $Je_1 = e_2, Je_2 = -e_1$ and $J\nu_3 = \nu_4, J\nu_4 = -\nu_3$. Furthermore, F can be viewed as a holomorphic line bundle with complex structure locally determined by $Jf^2 = f^3, Jf^3 = -f^2$ (see Subsection 4.2 for more details). As a consequence, it makes sense to talk about L^2 being **negative superminimal** in $\mathbb{R}^4$, which means that $A^{J\nu} = -JA^\nu$ holds for all normal vector fields $\nu \in \Gamma(NL)$, and about $\sigma \in \Gamma(F)$ being holomorphic. They proved the following result.

Theorem 3.9 ([KL12, Thm. 2, 3])**.**

1. *The submanifold $E + \sigma$ is associative in $\Lambda^2_-(T^*\mathbb{R}^4)$ if and only if L^2 is minimal in $\mathbb{R}^4$ and $\sigma \in \Gamma(F)$ is holomorphic.*
2. *The submanifold $\eta + F$ is coassociative in $\Lambda^2_-(T^*\mathbb{R}^4)$ if and only if L^2 is negative superminimal in $\mathbb{R}^4$ and $\eta \in \Gamma(E)$ is parallel with respect to the connection ∇^E on E induced by the Levi-Civita connection on $\mathbb{R}^4$.*

Proof (idea). 1: First, determine a basis for the tangent space to X^E_σ at every point $\omega \in \Phi(X^E_\sigma)$ via the immersion $\Phi : X^E_\sigma \to \Lambda^2_-(T^*\mathbb{R}^4)$. Identify $T_\omega(\Lambda^2_-(T^*\mathbb{R}^4)) \cong \operatorname{Im}\mathbb{O}$ and plug the basis vectors into the associator $[\cdot,\cdot,\cdot]$. Finally, establish the conditions under which that expression vanishes (cf. Proposition 2.22(iii)).

2: To begin with, find a basis for the tangent space to X^F_η at every point $\omega \in \Psi(X^F_\eta)$ using the immersion $\Psi : X^F_\eta \to \Lambda^2_-(T^*\mathbb{R}^4)$. Then determine the conditions under which the associative 3-form φ vanishes on X^F_η by plugging in the basis vectors (cf. Proposition 2.23(iv)). □

Remark 3.10. As $\sigma = 0$ and $\eta = 0$ are holomorphic and parallel, respectively, we can easily read off the result derived in [IKM05].

As in the special Lagrangian case, the idea arises to generalize this construction to other manifolds X^4, for which $M^7 = \Lambda^2_-(T^*X^4)$ still possesses a metric of holonomy in G_2. The first complete, noncompact examples of such metrics of holonomy equal to G_2 were constructed by Bryant and Salamon [BS89] for $X^4 = S^4, \mathbb{CP}^2$ with the standard metrics. Before stating the theorem, we clarify the setting.

We equip M^7 with the connection ∇ induced by the Levi-Civita connection on (X^4, g). This provides a canonical splitting of its tangent space $T_\omega M \cong \mathcal{H}_\omega \oplus \mathcal{V}_\omega$ into the horizontal and vertical spaces for any $\omega \in M$. We can identify $\mathcal{H}_\omega$ with the tangent space

$T_{\pi(\omega)}X$ of X via the linear isomorphism $\mathrm{Hor}_\omega = (\pi_*|_{\mathcal{H}_\omega})^{-1} : T_{\pi(\omega)}X \to \mathcal{H}_\omega$, where $\pi : M^7 = \Lambda^2_-(T^*X) \to X$ stands for the projection onto the base. On the other hand, $\mathcal{V}_\omega$ can be identified with the fiber $M_{\pi(\omega)} = \Lambda^2_-(T^*_{\pi(\omega)}X)$ through the linear isomorphism $\mathrm{Vert}_\omega : M_{\pi(\omega)} \to \mathcal{V}_\omega = T_\omega(M_{\pi(\omega)})$, $\sigma \mapsto \frac{d}{dt}(\omega + t\sigma)|_{t=0}$. This follows from the fact that $M_{\pi(\omega)}$ is a vector space as M is a vector bundle. (See [Wen22, Sec. 19.1] for more details.) Due to these identifications, the metric g on X induces metrics $g_{\mathcal{H}}$ and $g_{\mathcal{V}}$ with canonical volume forms $\mathrm{vol}_{\mathcal{H}}$ and $\mathrm{vol}_{\mathcal{V}}$ on $\mathcal{H}$ and $\mathcal{V}$, respectively.

Theorem 3.11 ([BS89, Thm. 4.1])**.** *Let (X^4, g) be either S^4 with the standard round metric or $\mathbb{CP}^2$ with the Fubini-Study metric, and let r denote the radial coordinate in the vertical fibers. Then there exist positive functions $u = u(r)$ and $v = v(r)$ such that*

$$g_{M^7} = u^2 g_{\mathcal{H}} \oplus v^2 g_{\mathcal{V}}$$

defines a complete metric on $M^7 = \Lambda^2_-(T^*X^4)$ with holonomy equal to G_2. Its fundamental 3-form φ is given by

$$\varphi = v^3 \,\mathrm{vol}_{\mathcal{V}} + u^2 v \, d\theta,$$

where $\theta_\omega = \pi^*\omega$ ($\omega \in M$) is the canonical soldering 2-form on M.

Let us now restrict the vector bundle $M^7 = \Lambda^2_-(T^*X^4) \to X$ to an oriented immersed submanifold $L^2 \subset X^4$ and fix some oriented local orthonormal adapted frame (e_1, e_2, ν_3, ν_4) along L with dual coframe (e^1, e^2, ν^3, ν^4). The anti-self-dual 2-forms $f^1 = e^1 \wedge e^2 - \nu^3 \wedge \nu^4$, $f^2 = e^1 \wedge \nu^3 - \nu^4 \wedge e^2$ and $f^3 = e^1 \wedge \nu^4 - e^2 \wedge \nu^3$ locally trivialize $\Lambda^2_-(T^*X)|_L$. We denote the horizontal lifts of the tangent and normal vectors in $TX|_L$ to $\mathcal{H}$ by $\bar{e}_i = \mathrm{Hor}\, e_i$, $\bar{\nu}_j = \mathrm{Hor}\, \nu_j$, $i = 1, 2$, $j = 3, 4$, and the vertical lifts of the anti-self-dual 2-forms on X to $\mathcal{V}$ by $\check{f}^k$, $k = 1, 2, 3$. Furthermore, we refer to their dual horizontal and vertical 1-forms as $\bar{e}^i, \bar{\nu}^j$ and $\check{f}_k$, respectively. The following diagram provides a compact illustration of the situation described for an $\omega \in \pi^{-1}(L) \subset M$:

$$\begin{array}{cccc}
 & \mathrm{span}\{\bar{e}_1, \bar{e}_2, \bar{\nu}_3, \bar{\nu}_4\}|_\omega & & \mathrm{span}\{\check{f}^1, \check{f}^2, \check{f}^3\}|_\omega \\
 & \| & & \| \\
T_\omega M = T_\omega\big(\Lambda^2_-(T^*X)\big) \cong & \mathcal{H}_\omega & \oplus & \mathcal{V}_\omega \\
 & \mathrm{Hor}_\omega \uparrow \cong & & \mathrm{Vert}_\omega \uparrow \cong \\
 & T_{\pi(\omega)}X & & M_{\pi(\omega)} = \Lambda^2_-(T^*_{\pi(\omega)}X) \\
 & \| & & \| \\
 & \mathrm{span}\{e_1, e_2, \nu_3, \nu_4\}|_{\pi(\omega)} & & \mathrm{span}\{f^1, f^2, f^3\}|_{\pi(\omega)}
\end{array}$$

As a result, the fundamental 3-form φ in Theorem 3.11 and its Hodge dual $\psi = *\varphi$ restricted to L are locally given by

$$\begin{aligned}\varphi = {} & v^3(\check{f}_1 \wedge \check{f}_2 \wedge \check{f}_3) + u^2 v\, \check{f}_1 \wedge (\bar{e}^1 \wedge \bar{e}^2 - \bar{\nu}^3 \wedge \bar{\nu}^4) \\ & + u^2 v\, \check{f}_2 \wedge (\bar{e}^1 \wedge \bar{\nu}^3 - \bar{\nu}^4 \wedge \bar{e}^2) + u^2 v\, \check{f}_3 \wedge (\bar{e}^1 \wedge \bar{\nu}^4 - \bar{e}^2 \wedge \bar{\nu}^3)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\psi = {} & u^4(\bar{e}^1 \wedge \bar{e}^2 \wedge \bar{\nu}^3 \wedge \bar{\nu}^4) - u^2 v^2\, \check{f}_2 \wedge \check{f}_3 \wedge (\bar{e}^1 \wedge \bar{e}^2 - \bar{\nu}^3 \wedge \bar{\nu}^4) \\ & - u^2 v^2\, \check{f}_3 \wedge \check{f}_1 \wedge (\bar{e}^1 \wedge \bar{\nu}^3 - \bar{\nu}^4 \wedge \bar{e}^2) - u^2 v^2\, \check{f}_1 \wedge \check{f}_2 \wedge (\bar{e}^1 \wedge \bar{\nu}^4 - \bar{e}^2 \wedge \bar{\nu}^3)\end{aligned} \tag{3.6}$$

[KM05, (17), (18)].

As before, f^1 is invariant under change of coordinates and thus globally defined on L (via $f^1 = \mathrm{vol}_L - *_{X^4}\mathrm{vol}_L$). Its span therefore defines a rank 1 bundle $E = \mathrm{span}\{f^1\}$ with orthogonal complement $F = E^\perp \overset{\mathrm{loc}}{=} \mathrm{span}\{f^2, f^3\}$ in $\Lambda^2_-(T^*X)|_L$. The total spaces of these bundles are 3- and 4-dimensional submanifolds of $M^7 = \Lambda^2_-(T^*X^4)$, respectively. Once again, we endow TL and NL with the natural complex structures locally given by $Je_1 = e_2, Je_2 = -e_1$ and $J\nu_3 = \nu_4, J\nu_4 = -\nu_3$. Karigiannis and Min-Oo proved the following result.

Theorem 3.12 ([KM05, Thm. 4.3])**.** *Let $X^4 = S^4$ or $\mathbb{CP}^2$. Then:*

1. *The submanifold E is associative in $\Lambda^2_-(T^*X^4)$ if and only if L^2 is minimal in X^4.*
2. *The submanifold F is coassociative in $\Lambda^2_-(T^*X^4)$ if and only if L^2 is negative superminimal in X^4.*

Proof (idea). 1: First, determine a basis for the tangent space to E at every point $\omega \in \Phi(E)$ via the immersion $\Phi : E \to \Lambda^2_-(T^*X)$. Then plug the three basis vectors into the coassociative 4-form ψ and establish the conditions under which the resulting 1-form vanishes (cf. Proposition 2.22(iv)).

2: To begin with, find a basis for the tangent space to F at every point $\omega \in \Psi(F)$ using the immersion $\Psi : F \to \Lambda^2_-(T^*X)$. Determine the conditions under which the associative 3-form φ vanishes on F by plugging in the basis vectors (cf. Proposition 2.23(iv)). □

We will answer the following remaining question in Subsection 4.2.

Question 3.13. Let $L^2 \subset X^4 = S^4, \mathbb{CP}^2$ be an oriented immersed submanifold and $\sigma \in \Gamma(F)$, $\eta \in \Gamma(E)$. Consider the spaces

$$X^E_\sigma = \big\{(x, \eta + \sigma_x) \in \Lambda^2_-(T^*X^4)|_L \,\big|\, x \in L,\, \eta \in E_x\big\} \qquad (\text{“}E + \sigma\text{”}),$$
$$X^F_\eta = \big\{(x, \eta_x + \sigma) \in \Lambda^2_-(T^*X^4)|_L \,\big|\, x \in L,\, \sigma \in F_x\big\} \qquad (\text{“}\eta + F\text{”}).$$

Under what conditions on L, σ and η are $E + \sigma$ and $\eta + F$ associative and coassociative in $\Lambda^2_-(T^*X^4)$, respectively?

3.4. Cayley submanifolds of $\$_-(\mathbb{R}^4)$ and $\$_-(S^4)$

We now come to the construction of Cayley submanifolds in the negative spinor bundle $\$_-(X)$, where X^4 is either Euclidean space $\mathbb{R}^4$ or the 4-dimensional sphere S^4. To begin with, let us briefly review some preliminaries discussed in [IKM05, Sec. 4.4] for $X = \mathbb{R}^4$ with the standard metric, which also hold true for S^4 with the standard round metric.

Let $e^1, \ldots, e^4$ be an orthonormal basis of T^*_xX at a fixed point $x \in X^4 = \mathbb{R}^4, S^4$. Then the Clifford algebra $\mathrm{Cl}(T^*_xX) \cong \mathrm{Cl}(T_xX)$ is generated by $e^1, \ldots, e^4$ subject to the relations

$$e^i \cdot e^j + e^j \cdot e^i = -2\delta^{ij}. \tag{3.7}$$

We write $\$_\pm$ for the ± 1 eigenspace of the pinor representation $\gamma(\lambda) \in \mathrm{End}(\$)$ of the volume element $\lambda = e^1 \cdot e^2 \cdot e^3 \cdot e^4 \in \mathrm{Cl}(T^*_xX)$. Both $\$_+$ and $\$_-$ are isomorphic to the quaternions

$\mathbb{H}$ (see Appendix B), and Clifford multiplication by a covector $\alpha \in T_x^*X$ interchanges them since $\lambda \cdot \alpha = -\alpha \cdot \lambda$. On the other hand, octonionic multiplication satisfies $u(uv) = u^2 v$ and $u_1(\bar{u}_2 v) = -u_2(\bar{u}_1 v)$ (2.4) for all $u, u_1, u_2, v \in \mathbb{O}$ with u_1 and u_2 orthogonal. Combining these two identities yields

$$u_i(u_j v) + u_j(u_i v) = -2\delta_{ij} v \tag{3.8}$$

for any orthonormal basis $u_1, \ldots, u_4$ of $\mathbb{H}e$ and any $v \in \mathbb{O}$. Furthermore, multiplication by elements in $\mathbb{H}e$ interchanges $\mathbb{H}$ and $\mathbb{H}e$ (see Appendix C). Comparing (3.7) and (3.8), we see that the pinor representation at each point $x \in X$ is obtained from octonionic multiplication by identifying $\$ = \$_+ \oplus \$_- \cong \mathbb{H}e \oplus \mathbb{H} \cong \mathbb{O}$ and $T_x^*X \cong \mathbb{H}e$. For covectors $\alpha \in T_x^*X$, we write it as

$$\gamma : T_x^*X \to \mathrm{End}(\$_+ \oplus \$_-), \quad \gamma(\alpha)(s) = \alpha s,$$

where the product stands for octonionic multiplication. When composing two elements of this representation, it is crucial to remember that $(\gamma(\alpha_1)\gamma(\alpha_2))(s) = \alpha_1(\alpha_2 s)$ is in general not equal to $(\alpha_1\alpha_2)s$ because $\mathbb{O}$ is not associative. For more details on spin geometry and representations, see Appendix B and, e.g., [Har90, Ch. 9–11], [Wen22, Sec. 50].

We now have the tools to review the bundle constructions in [IKM05] which provide candidates $V_\pm$ for Cayley submanifolds of $\$_-(\mathbb{R}^4)$. In this case, the ambient manifold is naturally isomorphic to $\mathbb{R}^8$ and possesses a canonical parallel Spin(7)-structure Φ [BS89].

Consider an oriented immersed submanifold $L^2 \subset \mathbb{R}^4$ and fix some oriented local orthonormal adapted frame (e_1, e_2, ν_3, ν_4) along L with dual coframe (e^1, e^2, ν^3, ν^4). Since e^1, e^2, ν^3, ν^4 are orthonormal, they satisfy $e^1 \cdot e^2 = \frac{1}{2} e^1 \wedge e^2$ and $\nu^3 \cdot \nu^4 = \frac{1}{2}\nu^3 \wedge \nu^4$ (see (B.1)), which implies that the terms $\gamma(e^1)\gamma(e^2) = \gamma(e^1 \cdot e^2)$ and $\gamma(\nu^3)\gamma(\nu^4) = \gamma(\nu^3 \cdot \nu^4)$ are independent of the choice of frame and hence globally defined. Let us now focus on a fixed point $x \in L$ and consider the restricted operators $\gamma(e^1)\gamma(e^2)$ and $\gamma(\nu^3)\gamma(\nu^4) : \$_- \to \$_-$. The identity $(ue)((ve)(wy)) = w((ue)((ve)y))$ for $u, v, w, y \in \mathbb{H}$, derived from (2.4), shows that both $\gamma(e^1)\gamma(e^2)$ and $\gamma(\nu^3)\gamma(\nu^4)$ are complex linear with respect to the natural complex structure $j_L = e^1 e^2 \in \{u \in \operatorname{Im}\mathbb{H} \mid |u| = 1\}$ on $\$_- \cong \mathbb{H}$. (A quick computation yields that j_L is independent of the choice of frame.) As $\gamma(e^1 \cdot e^2)^2 = \gamma(\nu^3 \cdot \nu^4)^2 = -1$, the two operators share the eigenvalues j_L and $-j_L$. Combining this with the fact that they are simultaneously diagonalizable because they commute, $\gamma(e^1)\gamma(e^2)$ and $\gamma(\nu^3)\gamma(\nu^4)$ differ at most by a sign. Due to the relation $\gamma(e^1 \cdot e^2)\gamma(\nu^3 \cdot \nu^4) = \gamma(\lambda) = \pm 1$ on $\$_\pm$, they must be equal on $\$_-$. Consequently, there exists a global canonical complex structure Γ on the restricted bundle $\$_-(\mathbb{R}^4)|_L$, locally given by $\Gamma = \gamma(e^1)\gamma(e^2) = \gamma(\nu^3)\gamma(\nu^4) : \$_- \to \$_-$. This operator provides a splitting of $\$_-(\mathbb{R}^4)|_L = V_+ \oplus V_-$ into the two eigenbundles $V_\pm$ of rank 2 corresponding to its eigenvalues $\pm j_L$. The total spaces of these bundles are 4-dimensional submanifolds of $\$_-(\mathbb{R}^4)$, and Ionel–Karigiannis–Min-Oo [IKM05] derived necessary and sufficient conditions on L^2 for $V_\pm$ to be Cayley in $\mathbb{R}^8$. As in the previous subsection, their findings are a special case of the "twisted" version in [KL12], so we go straight to the constructions therein.

Karigiannis and Leung examined the space

$$X_\psi = \left\{(x, \xi + \psi_x) \in \$_-(\mathbb{R}^4)|_L \,\middle|\, x \in L,\, \xi \in (V_+)_x\right\} \qquad (\text{"}V_+ + \psi\text{"})$$

for a section $\psi \in \Gamma(V_-)$. This is a "twisting" of the bundle V_+ over L obtained by affinely translating each fiber $(V_+)_x$ by a vector $\psi_x \in (V_-)_x$ in the orthogonal complement. Similarly to the G_2 case, $V_\pm$ can be viewed as a holomorphic line bundle with complex structures locally given by $Je^1 = e^2, Je^2 = -e^1$ on the base and $-\Gamma$ on the fiber (see Subsection 4.3 for more details). They proved the following result.

Theorem 3.14 ([KL12, Thm. 4]). *The submanifold $V_+ + \psi$ is Cayley in $\$_-(\mathbb{R}^4)$ if and only if L^2 is minimal in $\mathbb{R}^4$ and $\psi \in \Gamma(V_-)$ is holomorphic.*

Proof (idea). First, determine a basis for the tangent space to X_ψ at every point $s \in \Psi(X_\psi)$ via the immersion $\Psi : X_\psi \to \$_-(\mathbb{R}^4)$. Identify $T_s(\$_-(\mathbb{R}^4)) \cong \mathbb{O}$ and plug the basis vectors into the purely imaginary four-fold cross product $\mathrm{Im}(\cdot \times \cdot \times \cdot \times \cdot)$. Finally, establish the conditions under which that expression vanishes (cf. Proposition 2.27(iii)). $\square$

Remark 3.15. In the same way, we obtain an analogous result for $\chi + V_-$ with $\chi \in \Gamma(V_+)$.

Remark 3.16. As $\psi = 0$ is holomorphic, we can easily read off the result derived in [IKM05].

The process of generalizing this construction to other manifolds X^4, for which $M^8 = \$_-(X^4)$ still possesses a complete $\mathrm{Spin}(7)$-metric, starts again with a result by Bryant and Salamon, who constructed such a metric for $X^4 = S^4$ with the standard round metric g. Before stating the theorem, we clarify the setting.

As described in Appendix B, there exists a natural connection ∇, the spin connection, on M^8, induced by the Levi-Civita connection on (S^4, g). This provides a natural splitting of its tangent space $T_sM \cong \mathcal{H}_s \oplus \mathcal{V}_s$ into the horizontal and vertical spaces for any $s \in M$. We can identify $\mathcal{H}_s$ with the tangent space $T_{\pi(s)}S^4$ of S^4 via the linear isomorphism $\mathrm{Hor}_s = (\pi_*|_{\mathcal{H}_s})^{-1} : T_{\pi(s)}S^4 \to \mathcal{H}_s$, where $\pi : M^8 = \$_-(S^4) \to S^4$ stands for the projection onto the base. On the other hand, $\mathcal{V}_s$ can be identified with the fiber $M_{\pi(s)} = (\$_-)_{\pi(s)}(S^4)$ through the linear isomorphism $\mathrm{Vert}_s : M_{\pi(s)} \to \mathcal{V}_s = T_s(M_{\pi(s)})$, $\sigma \mapsto \frac{d}{dt}(s + t\sigma)|_{t=0}$. As in the G_2 case, this follows from the fact that $M_{\pi(s)}$ is a vector space. Due to these identifications, the metric g on S^4 induces metrics $g_\mathcal{H}$ and $g_\mathcal{V}$ with natural volume forms $\mathrm{vol}_\mathcal{H}$ and $\mathrm{vol}_\mathcal{V}$ on $\mathcal{H}$ and $\mathcal{V}$, respectively.

Theorem 3.17 ([BS89, Thm. 4.2], [Kar10, Sec. 2.2]). *Consider S^4 with the standard round metric and let r denote the radial coordinate in the vertical fibers. Then there exist positive functions $u = u(r)$ and $v = v(r)$ such that*

$$g_{M^8} = u^2 g_\mathcal{H} \oplus v^2 g_\mathcal{V}$$

defines a complete metric on $M^8 = \$_-(S^4)$ with holonomy equal to $\mathrm{Spin}(7)$. *Its fundamental 4-form Φ is given by*

$$\Phi = u^4 \,\mathrm{vol}_\mathcal{H} - u^2 v^2 (\omega_1 \wedge \sigma^1 + \omega_2 \wedge \sigma^2 + \omega_3 \wedge \sigma^3) + v^4 \,\mathrm{vol}_\mathcal{V},$$

where $\omega_1, \omega_2, \omega_3$ is an orthogonal basis of norm $\sqrt{2}$ for the self-dual 2-forms on $\mathcal{H}$ and $\sigma^1, \sigma^2, \sigma^3$ is the corresponding orthogonal basis for the self-dual 2-forms on $\mathcal{V}$.

Remark 3.18. The factor $\sqrt{2}$ in the preceding theorem appears due to our convention for the inner product on the exterior algebra $\Lambda^k V$ over some vector space V, namely $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)$ for $v_i, w_j \in V$ (see (B.2)).

Let us now restrict the vector bundle $M^8 = \slashed{S}_-(S^4) \to S^4$ to an oriented immersed submanifold $L^2 \subset S^4$ and fix some oriented local orthonormal adapted frame (e_1, e_2, ν_3, ν_4) along L with dual coframe (e^1, e^2, ν^3, ν^4). We denote its horizontal lift to $\mathcal{H}$ by $\bar{e}_i = \operatorname{Hor} e_i$, $\bar{\nu}_j = \operatorname{Hor} \nu_j$, and their dual horizontal 1-forms by $\bar{e}^i, \bar{\nu}^j$, $i = 1, 2$, $j = 3, 4$.

As in the case of $\slashed{S}_-(\mathbb{R}^4)$, there exists a natural complex structure Γ on the restricted bundle $\slashed{S}_-(S^4)|_L$, given by $\Gamma = \gamma(e^1)\gamma(e^2) = \gamma(\nu^3)\gamma(\nu^4)$. This operator provides a splitting of $\slashed{S}_-(S^4)|_L = V_+ \oplus V_-$ into the two eigenbundles $V_\pm$ of rank 2 corresponding to its eigenvalues $\pm j_L = \pm e^1 e^2$. Using similar reasoning as in the derivation of the equality $\gamma(e^1)\gamma(e^2) = \gamma(\nu^3)\gamma(\nu^4)$, we obtain

$$\gamma(e^1 \wedge e^2) = \gamma(\nu^3 \wedge \nu^4), \quad \gamma(e^1 \wedge \nu^3) = -\gamma(e^2 \wedge \nu^4), \quad \gamma(e^1 \wedge \nu^4) = \gamma(e^2 \wedge \nu^3). \tag{3.9}$$

Consider the standard basis

$$f^1 = e^1 \wedge e^2 + \nu^3 \wedge \nu^4, \quad f^2 = e^1 \wedge \nu^3 + \nu^4 \wedge e^2, \quad f^3 = e^1 \wedge \nu^4 + e^2 \wedge \nu^3$$

of self-dual 2-forms on S^4. By (3.9), it satisfies

$$\begin{aligned}
\gamma(f^1) &= \gamma(e^1 \wedge e^2) + \gamma(\nu^3 \wedge \nu^4) = 2\gamma(e^1 \wedge e^2) = 2\gamma(\nu^3 \wedge \nu^4) = 4\Gamma, \\
\gamma(f^2) &= \gamma(e^1 \wedge \nu^3) + \gamma(\nu^4 \wedge e^2) = 2\gamma(e^1 \wedge \nu^3) = -2\gamma(e^2 \wedge \nu^4), \\
\gamma(f^3) &= \gamma(e^1 \wedge \nu^4) + \gamma(e^2 \wedge \nu^3) = 2\gamma(e^1 \wedge \nu^4) = 2\gamma(e^2 \wedge \nu^3).
\end{aligned} \tag{3.10}$$

Using this, we compute that $\gamma(f^i)\gamma(f^j) = -\gamma(f^j)\gamma(f^i)$ for $i \neq j$, $\gamma(f^i)^2 = -16$ and

$$\begin{aligned}
\gamma(f^1)\gamma(f^2) &= 4\gamma(e^1 \wedge e^2)\gamma(e^1 \wedge \nu^3) = 8\gamma(e^2 \wedge \nu^3) = 4\gamma(f^3), \\
\gamma(f^1)\gamma(f^3) &= 4\gamma(e^1 \wedge e^2)\gamma(e^1 \wedge \nu^4) = 8\gamma(e^2 \wedge \nu^4) = -4\gamma(f^2), \\
\gamma(f^2)\gamma(f^3) &= 4\gamma(e^1 \wedge \nu^3)\gamma(e^1 \wedge \nu^4) = 8\gamma(\nu^3 \wedge \nu^4) = 4\gamma(f^1).
\end{aligned} \tag{3.11}$$

Now fix a local unit spinor s_1 in V_+. Then $\{s_1, s_2 = \frac{1}{4}\gamma(f^1)s_1 = \Gamma s_1 = j_L s_1\}$ and $\{s_3 = \frac{1}{4}\gamma(f^2)s_1, s_4 = \frac{1}{4}\gamma(f^3)s_1 = \Gamma s_3 = -j_L s_3\}$ form local orthonormal frames for V_+ and V_-, respectively. The latter follows from the fact that Γ anti-commutes with both $\gamma(f^2)$ and $\gamma(f^3)$, and that $\Gamma\gamma(f^2) = \gamma(f^3)$. We denote the vertical lifts of these spinors to $\mathcal{V}$ by $\check{s}_k$, with dual vertical 1-forms $\check{s}^k$, $k = 1, \dots, 4$. As in the G_2 case, we summarize the described setup using a diagram:

$$\begin{array}{cccc}
 & \operatorname{span}\{\bar{e}_1, \bar{e}_2, \bar{\nu}_3, \bar{\nu}_4\}|_s & & \operatorname{span}\{\check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4\}|_s \\
 & \| & & \| \\
T_s M = T_s\big(\slashed{S}_-(S^4)\big) \cong & \mathcal{H}_s & \oplus & \mathcal{V}_s \\
 & \uparrow \scriptstyle{\operatorname{Hor}_s}\, \cong & & \uparrow \scriptstyle{\operatorname{Vert}_s}\, \cong \\
 & T_{\pi(s)} S^4 & & M_{\pi(s)} = (\slashed{S}_-)_{\pi(s)}(S^4) \\
 & \| & & \| \\
 & \operatorname{span}\{e_1, e_2, \nu_3, \nu_4\}|_{\pi(s)} & & \operatorname{span}\{s_1, s_2, s_3, s_4\}|_{\pi(s)}
\end{array}$$

Consequently, the fundamental 4-form Φ in Theorem 3.17 restricted to L is locally given by

$$\begin{aligned}\Phi &= u^4\bar{e}^1\bar{e}^2\bar{\nu}^3\bar{\nu}^4 - u^2v^2(\omega_1\sigma^1 + \omega_2\sigma^2 + \omega_3\sigma^3) + v^4\check{s}^1\check{s}^2\check{s}^3\check{s}^4 \\ &= u^4\bar{e}^1\bar{e}^2\bar{\nu}^3\bar{\nu}^4 - u^2v^2(\bar{e}^1\bar{e}^2 + \bar{\nu}^3\bar{\nu}^4)(\check{s}^1\check{s}^2 + \check{s}^3\check{s}^4) - u^2v^2(\bar{e}^1\bar{\nu}^3 + \bar{\nu}^4\bar{e}^2)(\check{s}^1\check{s}^3 + \check{s}^4\check{s}^2) \\ &\quad - u^2v^2(\bar{e}^1\bar{\nu}^4 + \bar{e}^2\bar{\nu}^3)(\check{s}^1\check{s}^4 + \check{s}^2\check{s}^3) + v^4\check{s}^1\check{s}^2\check{s}^3\check{s}^4, \end{aligned} \tag{3.12}$$

where we omitted the wedge product symbols for clarity. Karigiannis and Min-Oo proved the following result.

Theorem 3.19 ([KM05, Thm. 4.8])**.** *The submanifold $V_\pm$ is Cayley in $\$_-(S^4)$ if and only if L^2 is minimal in S^4.*

Proof (idea). To begin with, determine a basis for the tangent space to $V_\pm$ at every point $s \in \Psi(V_\pm)$ via the immersion $\Psi : V_\pm \to \$_-(S^4)$. Establish the conditions under which the the rank 7 bundle valued 4-form η (see Proposition 2.27(iv)) vanishes on $V_\pm$ by plugging in the basis vectors. ☐

Remark 3.20. Contrary to [IKM05; KL12] and this thesis, the authors of [KM05] used the sign convention $\lambda = -e^1 \cdot e^2 \cdot \nu^3 \cdot \nu^4$ for the volume element. Due to this, they actually proved the above statement for the positive spinor bundle. The general idea remains the same but some adaptions are necessary to work out the statements and proof for the negative spinor bundle. In particular, the fundamental 4-form Φ_+ on $\$_+(S^4)$ differs by some signs from our formula for Φ on $\$_-(S^4)$ (see (3.12)):

$$\begin{aligned}\Phi_+ &= u^4\bar{e}^1\bar{e}^2\bar{\nu}^3\bar{\nu}^4 + u^2v^2(\bar{e}^1\bar{e}^2 - \bar{\nu}^3\bar{\nu}^4)(\check{s}^1\check{s}^2 - \check{s}^3\check{s}^4) + u^2v^2(\bar{e}^1\bar{\nu}^3 - \bar{\nu}^4\bar{e}^2)(\check{s}^1\check{s}^3 - \check{s}^4\check{s}^2) \\ &\quad + u^2v^2(\bar{e}^1\bar{\nu}^4 - \bar{e}^2\bar{\nu}^3)(\check{s}^1\check{s}^4 - \check{s}^2\check{s}^3) + v^4\check{s}^1\check{s}^2\check{s}^3\check{s}^4\end{aligned}$$

[Kar10, Sec. 2.2], [KM05, Thm. 4.5]. For the purpose of being coherent and sticking to one convention, we keep considering the negative spinor bundle, for which the above statement can be proved analogously to [KM05, Thm. 4.8].

As in the special Lagrangian and (co-)associative cases, we look at the following problem.

Question 3.21. Let $L^2 \subset S^4$ be an oriented immersed submanifold and $\psi \in \Gamma(V_-)$. Consider the space

$$X_\psi = \left\{(x, \xi + \psi_x) \in \$_-(S^4)|_L \,\middle|\, x \in L,\, \xi \in (V_+)_x\right\} \qquad (\text{“}V_+ + \psi\text{”}).$$

Under what conditions on L and ψ is $V_+ + \psi$ Cayley in $\$_-(S^4)$?

4. Analogous constructions for the Stenzel and Bryant–Salamon metrics

This section represents the core of this thesis and answers the three questions posed in Section 3. In other words, it provides necessary and sufficient conditions for the total spaces of the "twisted" bundles to be calibrated submanifolds in manifolds with special holonomy. This can be seen as a generalization of [KL12] to the case of complete, nonflat, noncompact manifolds.

4.1. Special Lagrangians in T^*S^n with the Stenzel metric

We first address Question 3.8. So let $L^q \subset S^n$ be an oriented immersed submanifold, $\mu \in \Omega^1(L)$ and X_μ ("$N^*L + \mu$") be the space obtained by affinely translating each fiber $N^*_x L$ by $\mu_x \in T^*_x L$. As before, S^n stands for the standard round sphere and we endow T^*S^n with the Calabi–Yau structure described in Subsection 3.2. Our goal is to find conditions on L and μ so that $N^*L + \mu$ is special Lagrangian in T^*S^n. We start with the conditions for it to be Lagrangian.

Theorem 4.1. *The submanifold $N^*L + \mu$ is Lagrangian in T^*S^n if and only if $\mu = 0$.*

Proof. Let $(e^1, \dots, e^q, \nu^{q+1}, \dots, \nu^n)$ be a local orthonormal adapted coframe along L. The immersion of $N^*L + \mu$ into T^*S^n, $\Phi : X_\mu \to T^*S^n$, is locally given by

$$\Phi : (u,t) \mapsto \Big(x(u), \sum_{k=q+1}^{n} t_k \nu^k(u) + \mu(u)\Big) = \Big(x(u), \sum_{k=q+1}^{n} t_k \nu^k(u) + \sum_{l=1}^{q} a_l(u) e^l(u)\Big),$$

where $u = (u_1, \dots, u_q)$ and $t = (t_{q+1}, \dots, t_n)$ are the coordinates on L and on the fiber, respectively, $x = (x_1, \dots, x_n)$ is the local immersion of L into S^n, and $a = (a_1, \dots, a_q)$ are the coordinates of μ with respect to the local trivialization $T^*L \overset{\text{loc}}{=} \operatorname{span}\{e^1, \dots, e^q\}$. For simplicity, let us omit the dependence on u in the following.

Define $\hat\nu(t) = \sum_{k=q+1}^{n} t_k \nu^k$ and $y(t) = |\hat\nu(t) + \mu|^2$. Since $e^1, \dots, e^q, \nu^{q+1}, \dots, \nu^n$ are orthonormal, we get

$$y(t) = |\hat\nu(t) + \mu|^2 = \Big| \sum_{k=q+1}^{n} t_k \nu^k + \sum_{l=1}^{q} a_l e^l \Big|^2 = \sum_{k=q+1}^{n} |t_k|^2 + \sum_{l=1}^{q} |a_l|^2 = |t|^2 + |a|^2. \tag{4.1}$$

Restricting the diffeomorphism $\Psi : T^*S^n \to Q$ (see (3.3)) to $\Phi(X_\mu) \subset T^*S^n$ gives

$$\Psi(x, \hat\nu + \mu) = x \cosh \sqrt{y} + i \frac{\hat\nu + \mu}{\sqrt{y}} \sinh \sqrt{y}, \tag{4.2}$$

where the second term is to be interpreted as 0 when $y = 0$.

To simplify the computations, let us modify the basis we are working with. Currently, we are writing everything in terms of $(x, e^1, \dots, e^q, \nu^{q+1}, \dots, \nu^n)$, which is an adapted orthonormal moving frame of $\mathbb{R}^{n+1}$ along L. We now fix a point $(u^*, t^*) \in X_\mu$ with $t^* \neq 0$. By imposing this additional condition, we guarantee that y vanishes nowhere in a sufficiently small neighborhood of (u^*, t^*). Let $(e_0^*, \dots, e_n^*)$ denote the orthonormal basis of

$\mathbb{R}^{n+1}$ given by the moving frame at that fixed point. Since Ψ is equivariant with respect to $\mathrm{SO}_{n+1}(\mathbb{R}) \subset \mathrm{O}_{n+1}(\mathbb{C})$, we can assume that $\hat{\nu}(u^*, t^*) = |t^*|\nu^{q+1}(u^*) = |t^*|e^*_{q+1} = t^*_{q+1}e^*_{q+1}$.

With respect to this basis, $\Phi(u^*, t^*)$ takes the form

$$\Phi(u^*, t^*) = (x, \hat{\nu} + \mu)(u^*, t^*) = \Big(e^*_0, |t^*|e^*_{q+1} + \sum_{l=1}^{q} a_l(u^*)e^*_l\Big).$$

Substituting this into (4.2), we obtain

$$z^* = (\Psi \circ \Phi)(u^*, t^*) = \cosh\sqrt{y(u^*, t^*)}\, e^*_0 + i\,\frac{|t^*|e^*_{q+1} + \sum_{l=1}^{q} a_l(u^*)e^*_l}{\sqrt{y(u^*, t^*)}} \sinh\sqrt{y(u^*, t^*)},$$

which simplifies to

$$z^* = \sum_{i=1}^{n} z^*_i e^*_i = \cosh\sqrt{y}\, e^*_0 + \sum_{l=1}^{q}\Big(i\,\frac{\sinh\sqrt{y}}{\sqrt{y}} a_l\Big)e^*_l + i\,\frac{\sinh\sqrt{y}}{\sqrt{y}}|t^*|e^*_{q+1},$$

omitting the dependence on (u^*, t^*). Therefore, the coefficients of the Kähler form ω_{St} corresponding to the Stenzel metric (see (3.4)) are given by

$$\begin{aligned}
a_{jk} &= \Big(\delta_{jk} + \frac{a_j a_k}{y}\tanh^2\sqrt{y}\Big)\, v' + \frac{4a_j a_k}{y}\sinh^2\sqrt{y}\, v'', \qquad j, k \le q,\\
a_{j,q+1} = a_{q+1,j} &= \frac{|t^*|a_j}{y}\tanh^2\sqrt{y}\, v' + \frac{4|t^*|a_j}{y}\sinh^2\sqrt{y}\, v'', \qquad j \le q,\\
a_{q+1,q+1} &= \Big(1 + \frac{|t^*|^2}{y}\tanh^2\sqrt{y}\Big)v' + \frac{4|t^*|^2}{y}\sinh^2\sqrt{y}\, v'',\\
a_{jk} &= \delta_{jk}\, v', \qquad j \text{ or } k \ge q+2
\end{aligned}$$

at the point z^*. Their symmetry allows us to write ω_{St} as

$$\begin{aligned}
\omega_{\mathrm{St}} = \frac{i}{2}\sum_{1 \le j \le k \le q} 2^{-\delta_{jk}} a_{jk}(dz_j \wedge d\bar{z}_k + dz_k \wedge d\bar{z}_j) + \frac{i}{2}\sum_{j=1}^{q} a_{j,q+1}(dz_j \wedge d\bar{z}_{q+1} + dz_{q+1} \wedge d\bar{z}_j)\\
+ \frac{i}{2}\, a_{q+1,q+1} dz_{q+1} \wedge d\bar{z}_{q+1} + \frac{i}{2}\sum_{j=q+2}^{n} v' dz_j \wedge d\bar{z}_j. \qquad (4.3)
\end{aligned}$$

We want to determine when ω_{St} vanishes on the tangent space to $N^*L + \mu$ at z^*. That space is spanned by $E_i = (\Psi \circ \Phi)_*(\partial_{u_i})$ and $F_j = (\Psi \circ \Phi)_*(\partial_{t_j})$ for $i = 1, \dots, q$ and $j = q+1, \dots, n$. Specifically, E_i is given by

$$\begin{aligned}
E_i = \Bigg[\frac{\sinh\sqrt{y}}{\sqrt{y}}\Big(\sum_{l=1}^{q} a_l \frac{\partial a_l}{\partial u_i}\Big)e^*_0 + \cosh\sqrt{y}\, e^*_i\\
+ i\,\frac{1}{y}\Big(\sum_{l=1}^{q} a_l \frac{\partial a_l}{\partial u_i}\Big)\Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}}\Big)(\hat{\nu} + \mu) + i\,\frac{\sinh\sqrt{y}}{\sqrt{y}}\,(\nabla_{e_i}\hat{\nu} + \nabla_{e_i}\mu)\Bigg]\Bigg|_{z^*}
\end{aligned}$$

for all $i = 1, \dots, q$. As we are working with normal coordinates, we can use (3.2) to compute

$$\begin{aligned}\nabla_{e_i}\hat{\nu} &= \nabla_{e_i}\Big(\sum_{k=q+1}^{n} t_k \nu^k\Big) = \sum_{k=q+1}^{n} t_k(\nabla_{e_i}\nu^k) = \sum_{k=q+1}^{n} t_k\Big(\sum_{l=1}^{q} A_{il}^k e^l\Big)\\ &= \sum_{l=1}^{q}\Big(\sum_{k=q+1}^{n} t_k A_{il}^k\Big)e^l = \sum_{l=1}^{q} A_{il}^{\hat{\nu}} e^l = \sum_{l=1}^{q} A_{il}^{\hat{\nu}} e_l^*\end{aligned}$$

and

$$\begin{aligned}\nabla_{e_i}\mu &= \nabla_{e_i}\Big(\sum_{l=1}^{q} a_l e^l\Big) = \sum_{l=1}^{q}\Big(\frac{\partial a_l}{\partial u_i}e^l + a_l\nabla_{e_i}e^l\Big) = \sum_{l=1}^{q}\Big(\frac{\partial a_l}{\partial u_i}e^l - \sum_{k=q+1}^{n} a_l A_{il}^k \nu^k\Big)\\ &= \sum_{l=1}^{q}\frac{\partial a_l}{\partial u_i}e^l - \sum_{k=q+1}^{n}\Big(\sum_{l=1}^{q} a_l A_{il}^k\Big)\nu^k = \sum_{l=1}^{q}\frac{\partial a_l}{\partial u_i}e_l^* - \sum_{k=q+1}^{n}\Big(\sum_{l=1}^{q} a_l A_{il}^k\Big)e_k^*\end{aligned}$$

at z^*. Thus, every E_i takes the form

$$\begin{aligned}E_i = {}& \frac{\sinh\sqrt{y}}{\sqrt{y}}\Big(\sum_{l=1}^{q} a_l\frac{\partial a_l}{\partial u_i}\Big)e_0^* + \cosh\sqrt{y}\, e_i^*\\ &+ i\,\frac{1}{y}\Big(\sum_{l=1}^{q} a_l\frac{\partial a_l}{\partial u_i}\Big)\Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}}\Big)\Big(|t^*|e_{q+1}^* + \sum_{l=1}^{q} a_l e_l^*\Big)\\ &+ i\,\frac{\sinh\sqrt{y}}{\sqrt{y}}\Bigg(\sum_{l=1}^{q}\Big(A_{il}^{\hat{\nu}} + \frac{\partial a_l}{\partial u_i}\Big)e_l^* - \sum_{k=q+1}^{n}\Big(\sum_{l=1}^{q} a_l A_{il}^k\Big)e_k^*\Bigg).\end{aligned}$$

On the other hand, $F_j = (\Psi\circ\Phi)_*(\partial_{t_j})$ is given by

$$F_j = \left[t_j\frac{\sinh\sqrt{y}}{\sqrt{y}}e_0^* + i\,\frac{t_j}{y}\Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}}\Big)(\hat{\nu}+\mu) + i\,\frac{\sinh\sqrt{y}}{\sqrt{y}}\nu^j\right]\Bigg|_{z^*}$$

for all $j = q+1, \dots, n$. That is,

$$\begin{aligned}F_{q+1} &= |t^*|\frac{\sinh\sqrt{y}}{\sqrt{y}}e_0^* + i\,\frac{|t^*|}{y}\Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}}\Big)\Big(|t^*|e_{q+1}^* + \sum_{l=1}^{q} a_l e_l^*\Big) + i\,\frac{\sinh\sqrt{y}}{\sqrt{y}}e_{q+1}^*\\ &= |t^*|\frac{\sinh\sqrt{y}}{\sqrt{y}}e_0^* + i\,\frac{1}{y}\Big(|t^*|^2\cosh\sqrt{y} + |a|^2\frac{\sinh\sqrt{y}}{\sqrt{y}}\Big)e_{q+1}^*\\ &\quad + i\,\frac{|t^*|}{y}\Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}}\Big)\sum_{l=1}^{q} a_l e_l^*,\\ F_j &= i\,\frac{\sinh\sqrt{y}}{\sqrt{y}}e_j^*, \qquad j = q+2, \dots, n,\end{aligned}$$

where we used $y = |t^*|^2 + |a|^2$ (4.1) to obtain the second line.

Our goal is to compute ω_{St} of any pair of vectors in the basis $E_1, \dots, E_q, F_{q+1}, \dots, F_n$. To do so, we first need to determine dz_j and $d\bar{z}_j$ of every basis vector for all $j = 1, \dots, n$. For every $i = 1, \dots, q$, we find

$$\begin{aligned} dz_j(E_i) &= \overline{d\bar{z}_j(E_i)} \\ &= \ \delta_{ij} \cosh\sqrt{y} \\ &\quad + i\Bigg(\frac{a_j}{y} \Big(\sum_{l=1}^{q} a_l \frac{\partial a_l}{\partial u_i} \Big) \Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) + \Big(A_{ij}^{\hat{\nu}} + \frac{\partial a_j}{\partial u_i} \Big) \frac{\sinh\sqrt{y}}{\sqrt{y}} \Bigg) \end{aligned}$$

for $j \le q$,

$$\begin{aligned} dz_{q+1}(E_i) &= -d\bar{z}_{q+1}(E_i) \\ &= i\Bigg(\frac{|t^*|}{y} \Big(\sum_{l=1}^{q} a_l \frac{\partial a_l}{\partial u_i} \Big) \Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) - \Big(\sum_{l=1}^{q} a_l A_{il}^{q+1} \Big) \frac{\sinh\sqrt{y}}{\sqrt{y}} \Bigg) \end{aligned}$$

for $j = q+1$, and

$$dz_j(E_i) = -d\bar{z}_j(E_i) = -i\Big(\sum_{l=1}^{q} a_l A_{il}^{j} \Big) \frac{\sinh\sqrt{y}}{\sqrt{y}}$$

for $j \ge q+2$. The remaining basis vectors satisfy

$$\begin{aligned} dz_j(F_{q+1}) &= -d\bar{z}_j(F_{q+1}) \\ &= \begin{cases} i\,\frac{|t^*|a_j}{y} \Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big), & j \le q, \\ i\,\frac{1}{y} \Big(|t^*|^2 \cosh\sqrt{y} + |a|^2 \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big), & j = q+1, \\ 0, & j \ge q+2, \end{cases} \end{aligned}$$

and

$$dz_j(F_i) = -d\bar{z}_j(F_i) = i\delta_{ij} \frac{\sinh\sqrt{y}}{\sqrt{y}}, \qquad i = q+2, \dots, n,\ j = 1, \dots, n.$$

We start with the pair (F_{q+1}, E_i) for $i = 1, \dots, q$. Let $j, k \in \{1, \dots, q\}$ be arbitrary. Using the above formulas, we compute

$$\begin{aligned} &(dz_j \wedge d\bar{z}_k + dz_k \wedge d\bar{z}_j)(F_{q+1}, E_i) \\ &\quad = i\,\frac{|t^*|}{y} \Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) \big(a_j (dz_k + d\bar{z}_k)(E_i) + a_k (dz_j + d\bar{z}_j)(E_i) \big) \\ &\quad = i\,\frac{|t^*|}{y} \Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) (2a_j \delta_{ki} + 2a_k \delta_{ji}) \cosh\sqrt{y}, \end{aligned}$$

$$\begin{aligned} &(dz_j \wedge d\bar{z}_{q+1} + dz_{q+1} \wedge d\bar{z}_j)(F_{q+1}, E_i) \\ &\quad = i\,\frac{|t^*|a_j}{y} \Big(\cosh\sqrt{y} - \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) (dz_{q+1} + d\bar{z}_{q+1})(E_i) \\ &\qquad + i\,\frac{1}{y} \Big(|t^*|^2 \cosh\sqrt{y} + |a|^2 \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) (dz_j + d\bar{z}_j)(E_i) \\ &\quad = 0 + i\,\frac{1}{y} \Big(|t^*|^2 \cosh\sqrt{y} + |a|^2 \frac{\sinh\sqrt{y}}{\sqrt{y}} \Big) (2\delta_{ij} \cosh\sqrt{y}), \end{aligned}$$

$$(dz_{q+1}\wedge d\bar z_{q+1})(F_{q+1},E_i)$$
$$= i\,\frac{1}{y}\left(|t^*|^2\cosh\sqrt{y}+|a|^2\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)(dz_{q+1}+d\bar z_{q+1})(E_i)=0.$$

For $j\geq q{+}2$, F_{q+1} satisfies $dz_j(F_{q+1})=d\bar z_j(F_{q+1})=0$, implying that $(dz_j\wedge d\bar z_j)(F_{q+1},E_i)=0$. Substituting these formulas into (4.3) yields

$$\begin{aligned}
\omega_{\mathrm{St}}(E_i,F_{q+1}) &= \frac{|t^*|\cosh\sqrt{y}}{y}\left(\cosh\sqrt{y}-\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)\sum_{j=1}^{q}a_j a_{ij}\\
&\quad + a_{i,q+1}\frac{\cosh\sqrt{y}}{y}\left(|t^*|^2\cosh\sqrt{y}+|a|^2\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)\\
&\quad +0+0\\
&= \frac{|t^*|\cosh\sqrt{y}}{y}\left(\cosh\sqrt{y}-\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)\\
&\quad \cdot\left(\sum_{j=1}^{q}\left(\left(a_j\delta_{ij}+\frac{a_i a_j^2}{y}\tanh^2\sqrt{y}\right)v'+\frac{4a_i a_j^2}{y}\sinh^2\sqrt{y}\;v''\right)\right)\\
&\quad +\frac{\cosh\sqrt{y}}{y}\left(|t^*|^2\cosh\sqrt{y}+|a|^2\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)\\
&\quad \cdot\left(\frac{|t^*|a_i}{y}\tanh^2\sqrt{y}\;v'+\frac{4|t^*|a_i}{y}\sinh^2\sqrt{y}\;v''\right)\\
&= a_i\frac{|t^*|\cosh\sqrt{y}}{y}\\
&\quad \cdot\Bigg[\left(\cosh\sqrt{y}-\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)\left(v'+\frac{|a|^2}{y}\left(\tanh^2\sqrt{y}\;v'+4\sinh^2\sqrt{y}\;v''\right)\right)\\
&\qquad +\left(|t^*|^2\cosh\sqrt{y}+|a|^2\frac{\sinh\sqrt{y}}{\sqrt{y}}\right)\frac{1}{y}\left(\tanh^2\sqrt{y}\;v'+4\sinh^2\sqrt{y}\;v''\right)\Bigg]\\
&= a_i\frac{|t^*|\cosh^2\sqrt{y}}{y}\left(\left(1-\frac{\tanh\sqrt{y}}{\sqrt{y}}+\tanh^2\sqrt{y}\right)v'+4\sinh^2\sqrt{y}\;v''\right)
\end{aligned}$$

for $i=1,\ldots,q$. Since t^* is nonzero by assumption, $|t^*|$ and y are positive, which further implies that $v',v''>0$ [Ste93, Prop. 6] and $1-\frac{\tanh\sqrt{y}}{\sqrt{y}}+\tanh^2\sqrt{y}>0$. Consequently, $\omega_{\mathrm{St}}(E_i,F_{q+1})$ vanishes if and only if $a_i=0$. In other words, we have $\omega_{\mathrm{St}}(E_i,F_{q+1})=0$ for all $i=1,\ldots,q$ if and only if $\mu(u^*)=\sum_{l=1}^{q}a_l(u^*)e_l^*=0$. (As $u^*\in L$ was arbitrary, $\mu=0\in\Omega^1(L)$ is a necessary condition for $N^*L+\mu$ to be Lagrangian. Thus, we could already conclude this proof by referring to the proof of [KM05, Thm. 3.1], but let us carry out the few remaining steps for the sake of completeness.)

As $\mu(u^*)=0$ is a necessary condition, it suffices to do the computations for the other pairs with the assumption that $a_i=0$ for $i=1,\ldots,q$. In that case, the Kähler form ω_{St} (4.3) simplifies to

$$\omega_{\mathrm{St}}=\frac{i}{2}\sum_{j=1}^{n}(v'dz_j\wedge d\bar z_j)+\frac{i}{2}\left(\tanh^2|t^*|\;v'+4\sinh^2|t^*|\;v''\right)dz_{q+1}\wedge d\bar z_{q+1}.$$

From this, we obtain

$$\omega_{\mathrm{St}}(F_j,\cdot) = -\frac{1}{2}v'\frac{\sinh|t^*|}{|t^*|}(dz_j + d\bar{z}_j)$$

for $j = q+2,\ldots,n$, which implies that $\omega_{\mathrm{St}}(F_j,E_i) = \omega_{\mathrm{St}}(F_j,F_k) = 0$ for all $i = 1,\ldots,q$ and $k = q+1,\ldots,n$. Lastly, we compute

$$\begin{aligned}\omega_{\mathrm{St}}(E_i,E_k) &= \frac{i}{2}v'\left(\cosh|t^*|(d\bar{z}_i - dz_i)(E_k) + i\,\frac{\sinh|t^*|}{|t^*|}\sum_{j=1}^{q} A^{\hat{\nu}}_{ij}(d\bar{z}_j + dz_j)(E_k)\right)\\ &= \frac{i}{2}u'\left(\cosh|t^*|\left(-2i\,\frac{\sinh|t^*|}{|t^*|}A^{\hat{\nu}}_{ki}\right) + i\,\frac{\sinh|t^*|}{|t^*|}\sum_{j=1}^{q} A^{\hat{\nu}}_{ij}\left(2\delta_{jk}\cosh|t^*|\right)\right)\\ &= u'\frac{\sinh|t^*|\cosh|t^*|}{|t^*|}(A^{\hat{\nu}}_{ki} - A^{\hat{\nu}}_{ik}) = 0\end{aligned}$$

for $i,k = 1,\ldots,q$. Consequently, ω_{St} vanishes on $T_{z^*}X_\mu$ if and only if $\mu(u^*) = 0$. Due to the smoothness of Φ,Ψ and ω, this equivalence holds not only for every $z = (\Psi\circ\Phi)(u,t)$ with $t \neq 0$, but also for arbitrary $(u,t) \in X_\mu$. Thus, X_μ is Lagrangian in T^*S^n if and only if $\mu = 0$. ☐

Unfortunately, Theorem 4.1 implies that twisting N^*L by 1-forms on L does not provide any new examples. In fact, all possible special Lagrangians constructed in this way were already described in Theorem 3.7. This shows that special Lagrangians of the form N^*L in T^*S^n are way more rigid than those in $T^*\mathbb{R}^n$. Nevertheless, let us capture this result in the following corollary.

Corollary 4.2. *The submanifold $N^*L+\mu$ is special Lagrangian in T^*S^n if and only if L^q is austere in S^n and $\mu = 0$.*

4.2. (Co-)associative submanifolds of $\Lambda^2_-(T^*X)$ with the Bryant–Salamon metric for $X^4 = S^4, \mathbb{CP}^2$

Next, we discuss Question 3.13. Let $L^2 \subset X^4 = S^4, \mathbb{CP}^2$ be an oriented immersed submanifold, and $\eta \in \Gamma(E)$ and $\sigma \in \Gamma(F)$ be sections of the bundle $E = \mathrm{span}\{f^1\}$ and its orthogonal complement $F = E^\perp \overset{\mathrm{loc}}{=} \mathrm{span}\{f^2,f^3\}$ in $\Lambda^2_-(T^*X)|_L$. As before, S^4 and $\mathbb{CP}^2$ are equipped with the standard round metric and the Fubini-Study metric, respectively, and we endow $M^7 = \Lambda^2_-(T^*X^4)$ with the Bryant–Salamon metric of holonomy G_2 (see Theorem 3.11). We examine the spaces X^E_σ ("$E+\sigma$") and X^F_η ("$\eta+F$") obtained by affinely translating each fiber E_x and F_x by $\sigma_x \in F_x$ and $\eta_x \in E_x$, respectively. Our goal is to determine necessary and sufficient conditions on L, σ and η so that $E+\sigma$ and $\eta+F$ are associative and coassociative in $M^7 = \Lambda^2_-(T^*X^4)$. We begin by establishing some needed formulas and constructing a holomorphic structure on F.

Lemma 4.3. *Let ∇ denote the connection on $\Lambda^2_-(T^*X)|_L$ induced by the Levi-Civita connection on X^4. Using normal coordinates (3.1) at any $u^* \in L$ yields*

$$\begin{aligned}
\nabla_{e_i} f^1 &= (A^4_{i1} - A^3_{i2}) f^2 + (-A^3_{i1} - A^4_{i2}) f^3, \\
\nabla_{e_i} f^2 &= (A^3_{i2} - A^4_{i1}) f^1, \\
\nabla_{e_i} f^3 &= (A^4_{i2} + A^3_{i1}) f^1
\end{aligned}$$

at that point for $i = 1, 2$.

Proof. All three expressions are obtained by using the Leibniz rule for the covariant derivative and then applying the identities (3.2) [IKM05, Proof of Prop. 4.1.1]. We demonstrate this for the second one, the other two are derived similarly. We compute

$$\begin{aligned}
\nabla_{e_i} f^2 &= \nabla_{e_i}(e^1 \wedge \nu^3 - \nu^4 \wedge e^2) \\
&= (\nabla_{e_i} e^1) \wedge \nu^3 + e^1 \wedge (\nabla_{e_i} \nu^3) - (\nabla_{e_i} \nu^4) \wedge e^2 - \nu^4 \wedge (\nabla_{e_i} e^2) \\
&= (-A^3_{i1} \nu^3 - A^4_{i1} \nu^4) \wedge \nu^3 + e^1 \wedge (A^3_{i1} e^1 + A^3_{i2} e^2) \\
&\quad - (A^4_{i1} e^1 + A^4_{i2} e^2) \wedge e^2 - \nu^4 \wedge (-A^3_{i2} \nu^3 - A^4_{i2} \nu^4) \\
&= (A^3_{i2} - A^4_{i1})(e^1 \wedge e^2 - \nu^3 \wedge \nu^4) \\
&= (A^3_{i2} - A^4_{i1}) f^1
\end{aligned}$$

at u^* for $i = 1, 2$. □

With L being a 2-dimensional oriented immersed submanifold of X^4, the tangent bundle $TL \overset{\text{loc}}{=} \text{span}\{e_1, e_2\}$ and normal bundle $NL \overset{\text{loc}}{=} \text{span}\{\nu_3, \nu_4\}$ can be endowed with natural almost complex structures J_T and J_N, locally given by $J_T e_1 = e_2, J_T e_2 = -e_1$ and $J_N \nu_3 = \nu_4, J_N \nu_4 = -\nu_3$. As TL and NL are of rank 2, their Nijenhuis tensors vanish automatically, turning J_T and J_N into complex structures. Let g_L denote the metric on L induced by the metric on X. It is easy to check that it satisfies $g_L(\cdot,\cdot) = g_L(J_T\,\cdot, J_T\,\cdot)$, which allows us to view L as a complex manifold of dimension 1 with Hermitian metric g_L. Furthermore, all k-forms on L are trivial for $k > 3$, implying that its associated 2-form $\omega_L(\cdot,\cdot) = g_L(J_T\,\cdot,\cdot)$ must be closed. Hence, (L, g_L, J_T, ω_L) forms a complex 1-dimensional Kähler manifold. Similarly, the rank 2 vector bundle $F \overset{\text{loc}}{=} \text{span}\{f^2, f^3\}$ can be endowed with a complex structure J_F locally given by $J_F f^2 = f^3$ and $J_F f^3 = -f^2$. This turns $\{f^2\}$ into a local complex frame for F and, consequently, F into a rank 1 complex vector bundle over L.

The Levi-Civita connection on X induces connections ∇ on $\Lambda^2_-(T^*X)|_L$ and $\nabla^F = \pi_F \circ \nabla$ on F, where $\pi_F : \Lambda^2_-(T^*X)|_L = E \oplus F \to F$ stands for the projection onto F. The preceding lemma combined with the fact that f^1 is orthogonal to F then shows that $\nabla^F_{e_i} f^k = \pi_F(\nabla_{e_i} f^k) = 0$ for $i = 1, 2$ and $k = 2, 3$ at any $u^* \in L$, using normal coordinates at that point. From this, we deduce that $(\nabla^F_{e_i} J_F)(f^k) = \nabla^F_{e_i}(J_F f^k) - J_F(\nabla^F_{e_i} f^k) = 0$, which generalizes to $\nabla^F J_F = 0$ as the latter is a tensor. Consequently, the connection ∇^F is complex linear with respect to J_F, which allows us to define the operator $\bar\partial_F \overset{\text{def}}{=} (\nabla^F)^{0,1} : \Omega^{r,s}(F) \to \Omega^{r,s+1}(F)$. As $\bar\partial_F$ satisfies the Leibniz rule [Huy05, p. 176], it defines a pseudo-holomorphic structure on F. However, since L has complex dimension 1, there exist no $(0,2)$-forms on F. Thus, $\bar\partial_F^2 : \Omega^{r,s}(F) \to \Omega^{r,s+2}(F) = \{0\}$ must be trivial, implying that $\bar{\partial}_F = (\nabla^F)^{0,1}$ actually defines a holomorphic structure on F, turning it into a holomorphic vector bundle. Then a section $\sigma \in \Gamma(F)$ is holomorphic if $\bar{\partial}_F\sigma = (\nabla^F)^{0,1}\sigma = 0$. See [Joy07, Ch. 5] and [Mor07, Ch. 9] for background on Kähler manifolds and holomorphic vector bundles.

Theorem 4.4. *Let $X^4 = S^4$ or $\mathbb{CP}^2$. Then:*

1. *The submanifold $E + \sigma$ is associative in $\Lambda^2_-(T^*X^4)$ if and only if L^2 is minimal in X^4 and $\sigma \in \Gamma(F)$ is holomorphic.*
2. *The submanifold $\eta + F$ is coassociative in $\Lambda^2_-(T^*X^4)$ if and only if L^2 is negative superminimal in X^4 and $\eta \in \Gamma(E)$ is parallel with respect to the induced connection ∇^E on E from the Levi-Civita connection on X^4.*

Remark 4.5. Note that the conditions on L, σ and η are the same as in the case of $\mathbb{R}^4$ in [KL12] (see Theorem 3.9).

We split the proof into two parts with the purpose of making it easier to follow.

Proof of 1. The immersion of $E + \sigma$ into $M^7 = \Lambda^2_-(T^*X^4)$, $\Phi : X^E_\sigma \to M$, is locally given by

$$\Phi : (u, t_1) \mapsto \big(x(u), t_1 f^1(u) + \sigma(u)\big) = \big(x(u), t_1 f^1(u) + a(u)f^2(u) + b(u)f^3(u)\big),$$

where $u = (u_1, u_2)$ and t_1 are the coordinates on L and on the fiber, respectively, $x = (x_1, \ldots, x_4)$ is the local immersion of L into X^4, and a and b are the coordinates of σ with respect to the local trivialization $F \overset{\text{loc}}{=} \operatorname{span}\{f^2, f^3\}$. Then the tangent space to X^E_σ at some fixed point $\omega^* = \Phi(u^*, t_1^*) \in M$ is spanned by $E_1 = \Phi_*(\partial_{u_1})$, $E_2 = \Phi_*(\partial_{u_2})$ and $F_1 = \Phi_*(\partial_{t_1})$, omitting the dependence on (u^*, t^*). By Lemma 4.3, we have

$$\begin{aligned}
\mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^1) &= \mathrm{Vert}_{\omega^*}\big((A^4_{i1} - A^3_{i2})f^2 + (-A^3_{i1} - A^4_{i2})f^3\big)\\
&= (A^4_{i1} - A^3_{i2})\check{f}^2 + (-A^3_{i1} - A^4_{i2})\check{f}^3,\\
\mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^2) &= \mathrm{Vert}_{\omega^*}\big((A^3_{i2} - A^4_{i1})f^1\big) = (A^3_{i2} - A^4_{i1})\check{f}^1,\\
\mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^3) &= \mathrm{Vert}_{\omega^*}\big((A^4_{i2} + A^3_{i1})f^1\big) = (A^4_{i2} + A^3_{i1})\check{f}^1
\end{aligned}$$

for $i = 1, 2$. Using these formulas, we find that E_i takes the form

$$\begin{aligned}
E_i &= \bar{e}_i + \mathrm{Vert}_{\omega^*}\big(t_1^*\nabla_{e_i} f^1 + \nabla_{e_i}(af^2 + bf^3)\big)\\
&= \bar{e}_i + \mathrm{Vert}_{\omega^*}(t_1^*\nabla_{e_i} f^1 + a\nabla_{e_i} f^2 + b\nabla_{e_i} f^3 + a_i f^2 + b_i f^3)\\
&= \bar{e}_i + t_1^*\mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^1) + a\mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^2) + b\mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^3) + a_i\check{f}^2 + b_i\check{f}^3\\
&= \bar{e}_i + A_i\check{f}^1 + B_i\check{f}^2 + C_i\check{f}^3
\end{aligned}$$

with

$$A_i = a(A^3_{i2} - A^4_{i1}) + b(A^4_{i2} + A^3_{i1}), \quad B_i = t_1^*(A^4_{i1} - A^3_{i2}) + a_i, \quad C_i = t_1^*(-A^3_{i1} - A^4_{i2}) + b_i,$$

and $a_i = \frac{\partial a}{\partial u_i}$, $b_i = \frac{\partial b}{\partial u_i}$ for $i = 1, 2$. Lastly, the third basis vector is given by

$$F_1 = \Phi_*\left(\frac{\partial}{\partial t_1}\right) = \check{f}^1.$$

Next, we determine when $E_2 \lrcorner E_1 \lrcorner F_1 \lrcorner \psi$ vanishes (cf. Proposition 2.22(iv)). Using the formula

$$\begin{aligned}\psi = u^4(\bar{e}^1 \wedge \bar{e}^2 \wedge \bar{\nu}^3 \wedge \bar{\nu}^4) - u^2v^2\, \check{f}_2 \wedge \check{f}_3 \wedge (\bar{e}^1 \wedge \bar{e}^2 - \bar{\nu}^3 \wedge \bar{\nu}^4) \\ - u^2v^2\, \check{f}_3 \wedge \check{f}_1 \wedge (\bar{e}^1 \wedge \bar{\nu}^3 - \bar{\nu}^4 \wedge \bar{e}^2) - u^2v^2\, \check{f}_1 \wedge \check{f}_2 \wedge (\bar{e}^1 \wedge \bar{\nu}^4 - \bar{e}^2 \wedge \bar{\nu}^3)\end{aligned}$$

(see (3.6)), we compute

$$\begin{aligned}F_1 \lrcorner \psi &= -u^2v^2\big(-\check{f}_3 \wedge (\bar{e}^1 \wedge \bar{\nu}^3 - \bar{\nu}^4 \wedge \bar{e}^2) + \check{f}_2 \wedge (\bar{e}^1 \wedge \bar{\nu}^4 - \bar{e}^2 \wedge \bar{\nu}^3)\big), \\ E_1 \lrcorner F_1 \lrcorner \psi &= -u^2v^2(\check{f}_3 \wedge \bar{\nu}^3 - \check{f}_2 \wedge \bar{\nu}^4) \\ &\quad - u^2v^2\big(-C_1(\bar{e}^1 \wedge \bar{\nu}^3 - \bar{\nu}^4 \wedge \bar{e}^2) + B_1(\bar{e}^1 \wedge \bar{\nu}^4 - \bar{e}^2 \wedge \bar{\nu}^3)\big), \\ E_2 \lrcorner E_1 \lrcorner F_1 \lrcorner \psi &= -u^2v^2(C_2\bar{\nu}^3 - B_2\bar{\nu}^4) - u^2v^2(-C_1\bar{\nu}^4 - B_1\bar{\nu}^3) \\ &= -u^2v^2\big((-B_1 + C_2)\bar{\nu}^3 + (-B_2 - C_1)\bar{\nu}^4\big).\end{aligned}$$

Since $u, v > 0$, $E_2 \lrcorner E_1 \lrcorner F_1 \lrcorner \psi$ equals zero if and only if the equations

$$\begin{aligned}0 = -B_1 + C_2 &= -t_1^*(A_{11}^4 - A_{12}^3) - a_1 + t_1^*(-A_{12}^3 - A_{22}^4) + b_2 \\ &= -t_1^*(A_{11}^4 + A_{22}^4) + (-a_1 + b_2)\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned}0 = \quad B_2 + C_1 &= \quad t_1^*(A_{12}^4 - A_{22}^3) + a_2 + t_1^*(-A_{11}^3 - A_{12}^4) + b_1 \\ &= -t_1^*(A_{11}^3 + A_{22}^3) + (a_2 + b_1)\end{aligned} \tag{4.5}$$

are satisfied. Given that $(u^*, t_1^*) \in X_\sigma^E$ was arbitrary, X_σ^E is associative in M^7 if and only if (4.4) and (4.5) hold true on all of X_σ^E. This is equivalent to the conditions

(I) $\operatorname{Tr} A^3 = A_{11}^3 + A_{22}^3 = 0$ and $\operatorname{Tr} A^4 = A_{11}^4 + A_{22}^4 = 0$, or, in other words, L is minimal in X,

(II) $a_1 = b_2$ and $a_2 = -b_1$.

It remains to show that (II) is equivalent to $\sigma \in \Gamma(F)$ being holomorphic. Recall that σ is locally given by $\sigma = af^2 + bf^3$ and that it is holomorphic if $\bar{\partial}_F\sigma = (\nabla^F)^{0,1}\sigma = 0$. Since $e_1 + ie_2$ locally trivializes $(TL)^{0,1}$, this is equivalent to $\nabla^F_{e_1+ie_2}\sigma = 0$. We compute

$$\begin{aligned}\nabla^F_{e_1+ie_2}\sigma &= \nabla^F_{e_1}\sigma + J_F(\nabla^F_{e_2}\sigma) \\ &= \pi_F(\nabla_{e_1}\sigma) + J_F\big(\pi_F(\nabla_{e_2}\sigma)\big) \\ &= \pi_F(a\nabla_{e_1}f^2 + b\nabla_{e_1}f^3 + a_1f^2 + b_1f^3) \\ &\quad + J_F\big(\pi_F(a\nabla_{e_2}f^2 + b\nabla_{e_2}f^3 + a_2f^2 + b_2f^3)\big) \\ &= a_1f^2 + b_1f^3 + J_F(a_2f^2 + b_2f^3) \\ &= (a_1 - b_2)f^2 + (a_2 + b_1)f^3,\end{aligned}$$

where we used Lemma 4.3 and the fact that f^1 is orthogonal to F. Consequently, σ is holomorphic if and only if (II) holds, which concludes the proof. □

We proceed to the proof of the second statement.

Proof of 2. The immersion of $\eta + F$ into M, $\Psi : X_\eta^F \to M$, is locally given by

$$\Psi : (u,t) \mapsto \big(x(u), t_2 f^3(u) + t_3 f^3(u) + \eta(u)\big) = \big(x(u), t_2 f^3(u) + t_3 f^3(u) + \gamma(u) f^1(u)\big),$$

where $u = (u_1, u_2)$ and $t = (t_2, t_3)$ are the coordinates on L and on the fiber, respectively, $x = (x_1, \dots, x_4)$ is the local immersion of L into X^4 and $\gamma \in C^\infty(L)$ is the globally defined function such that $\eta = \gamma f^1$. Then the tangent space to X_η at some fixed point $\omega^* = \Psi(u^*, t^*) \in M$ is spanned by $E_1 = \Psi_*(\partial_{u_1})$, $E_2 = \Psi_*(\partial_{u_2})$, $F_2 = \Psi_*(\partial_{t_2})$ and $F_3 = \Psi_*(\partial_{t_3})$, omitting the dependence on (u^*, t^*). As in the proof of the first statement, we use Lemma 4.3 to compute E_i, and we find that

$$\begin{aligned} E_i &= \bar{e}_i + \mathrm{Vert}_{\omega^*}\big(t_2^* \nabla_{e_i} f^2 + t_3^* \nabla_{e_i} f^3 + \nabla_{e_i}(\gamma f^1)\big) \\ &= \bar{e}_i + t_2^* \mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^2) + t_3^* \mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^3) + \gamma \mathrm{Vert}_{\omega^*}(\nabla_{e_i} f^1) + \gamma_i \check{f}^1 \\ &= \bar{e}_i + A_i \check{f}^1 + B_i \check{f}^2 + C_i \check{f}^3 \end{aligned}$$

with

$$A_i = t_2^*(A_{i2}^3 - A_{i1}^4) + t_3^*(A_{i2}^4 + A_{i1}^3) + \gamma_i, \quad B_i = \gamma(A_{i1}^4 - A_{i2}^3), \quad C_i = \gamma(-A_{i1}^3 - A_{i2}^4),$$

and $\gamma_i = \frac{\partial \gamma}{\partial u_i}$ for $i = 1, 2$. Furthermore, F_2 and F_3 are given by

$$F_j = \Psi_*\left(\frac{\partial}{\partial t_j}\right) = \check{f}^j, \qquad j = 2, 3.$$

By Proposition 2.23, X_η is coassociative in M if and only if $\varphi|_{X_\eta} = 0$. Using the formula

$$\begin{aligned} \varphi &= v^3(\check{f}_1 \wedge \check{f}_2 \wedge \check{f}_3) + u^2 v\, \check{f}_1 \wedge (\bar{e}^1 \wedge \bar{e}^2 - \bar{\nu}^3 \wedge \bar{\nu}^4) \\ &\quad + u^2 v\, \check{f}_2 \wedge (\bar{e}^1 \wedge \bar{\nu}^3 - \bar{\nu}^4 \wedge \bar{e}^2) + u^2 v\, \check{f}_3 \wedge (\bar{e}^1 \wedge \bar{\nu}^4 - \bar{e}^2 \wedge \bar{\nu}^3) \end{aligned}$$

(see (3.5)), we obtain $\varphi(F_2, F_3, \cdot) = v^3 \check{f}_1$, which implies $\varphi(F_2, F_3, E_i) = v^3 A_i$ for $i = 1, 2$. On the other hand,

$$\begin{aligned} \varphi(E_1, E_2, \cdot) &= u^2 v \check{f}_1 + v^3\big((A_1 B_2 - A_2 B_1)\check{f}_3 + (-A_1 C_2 + A_2 C_1)\check{f}_2 + (B_1 C_2 - B_2 C_1)\check{f}_1\big) \\ &\quad - u^2 v(A_2 \bar{e}^2 + B_2 \bar{\nu}^3 + C_2 \bar{\nu}^4) + u^2 v(-A_1 \bar{e}^1 + B_1 \bar{\nu}^4 - C_1 \bar{\nu}^3) \\ &= \big(u^2 v + v^3(B_1 C_2 - B_2 C_1)\big)\check{f}_1 + v^3(-A_1 C_2 + A_2 C_1)\check{f}_2 + v^3(A_1 B_2 - A_2 B_1)\check{f}_3 \\ &\quad - u^2 v\big(A_1 \bar{e}^1 + A_2 \bar{e}^2 + (B_2 + C_1)\bar{\nu}^3 + (-B_1 + C_2)\bar{\nu}^4\big) \end{aligned}$$

yields $\varphi(E_1, E_2, F_2) = v^3(-A_1 C_2 + A_2 C_1)$ and $\varphi(E_1, E_2, F_3) = v^3(A_1 B_2 - A_2 B_1)$. Since $v > 0$, we deduce that φ vanishes on $T_{\omega^*} X_\eta$ if and only if the conditions

$$A_1 = A_2 = 0, \quad -A_1 C_2 + A_2 C_1 = 0 \quad \text{and} \quad A_1 B_2 - A_2 B_1 = 0$$

are satisfied. As the last two follow from the first, this is equivalent to $A_i = 0$ for $i = 1, 2$.

Since $(u^*, t^*) \in X_\eta$ was arbitrary, X_η is coassociative in M if and only if $A_i(t) = t_2(A_{i2}^3 - A_{i1}^4) + t_3(A_{i2}^4 + A_{i1}^3) + \gamma_i$ vanishes at every point $(u, t) \in X_\eta$ for $i = 1, 2$, omitting the dependence on u. Define $\nu(t) = t_2\nu_3 + t_3\nu_4$ and $\nu^\perp(t) = J_N\nu(t) = -t_3\nu_3 + t_2\nu_4$. Then A_i takes the form $A_i = A^\nu_{i2} - A^{\nu^\perp}_{i1} + \gamma_i$ for $i = 1, 2$. From the relations

$$A_i(\lambda t) = A^{\lambda\nu(t)}_{i2} - A^{\lambda\nu^\perp(t)}_{i1} + \gamma_i = \lambda(A^{\nu(t)}_{i2} - A^{\nu^\perp(t)}_{i1}) + \gamma_i,$$
$$A_i(\lambda t^\perp) = A^{\lambda\nu(t^\perp)}_{i2} - A^{\lambda\nu^\perp(t^\perp)}_{i1} + \gamma_i = \lambda(A^{\nu^\perp(t)}_{i2} + A^{\nu(t)}_{i1}) + \gamma_i$$

for $(t_2, t_3)^\perp = (-t_3, t_2)$ and $\lambda \in \mathbb{R}$, we deduce that A_i vanishes on all of X_η if and only if

$$A^\nu_{i2} - A^{\nu^\perp}_{i1} = 0, \quad A^{\nu^\perp}_{i2} + A^\nu_{i1} = 0 \quad \text{and} \quad \gamma_i = 0.$$

Combining these conditions yields

$$A^{J_N\nu} = A^{\nu^\perp} = \begin{pmatrix} A^{\nu^\perp}_{11} & A^{\nu^\perp}_{12} \\ A^{\nu^\perp}_{12} & A^{\nu^\perp}_{22} \end{pmatrix} = \begin{pmatrix} A^\nu_{12} & A^\nu_{22} \\ -A^\nu_{11} & -A^\nu_{12} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^\nu_{11} & A^\nu_{12} \\ A^\nu_{12} & A^\nu_{22} \end{pmatrix} = -J_T A^\nu$$

and $\gamma_1 = \gamma_2 = 0$. Since $\nu(t)$ and $\nu^\perp(t)$ form a local frame for NL whenever $t \neq 0$, this is equivalent to L being negative superminimal and $\gamma \in C^\infty(L)$ being constant.

From Lemma 4.3, we know that $\nabla_{e_i} f^1$ is orthogonal to E. Using this fact, we compute

$$\nabla^E_{e_i}\eta = \pi_E(\nabla_{e_i}\eta) = \pi_E(\gamma_i f^1 + \gamma\nabla_{e_i} f^1) = \gamma_i f^1$$

for $i = 1, 2$, where $\pi_E : E \oplus F \to E$ stands for the projection onto E. This shows that γ is constant if and only if η is parallel with respect to ∇^E. Thus, X_η is coassociative in M if and only if L is negative superminimal in X and η is parallel with respect to ∇^E. ☐

4.3. Cayley submanifolds of $\$_-(S^4)$ with the Bryant–Salamon metric

Lastly, we turn to Question 3.21. Let $L^2 \subset S^4$ be an oriented immersed submanifold and $\psi \in \Gamma(V_-)$ be a section of the bundle $V_- \overset{\text{loc}}{=} \operatorname{span}\{s_3, s_4\}$. As before, S^4 carries the standard round metric, while $M^8 = \$_-(S^4)$ is endowed with the Bryant–Salamon metric of holonomy Spin(7) (see Theorem 3.17). We consider the space X_ψ ("$V_+ + \psi$") formed by affinely translating each fiber $(V_+)_x$ of the bundle $V_+ \overset{\text{loc}}{=} \operatorname{span}\{s_1, s_2\}$ by $\psi_x \in (V_-)_x$. Our goal is to determine necessary and sufficient conditions on L and ψ for $V_+ + \psi$ to be Cayley in $\$_-(S^4)$. We start with a lemma and the construction of a holomorphic structure on $V_\pm$.

Lemma 4.6 ([KM05, Sec. 4.2])**.** *Let ∇ denote the connection on $\$_-(S^4)|_L$ induced by the Levi-Civita connection on S^4. Then $\nabla_{e_i}\Gamma$ interchanges V_+ and V_-, and every local section s of $V_\pm$ satisfies*

$$\nabla_{e_i} s = \mp\frac{1}{2} j_L(\nabla_{e_i}\Gamma)s$$

for $i = 1, 2$.

Proof. To simplify notation, let us focus on a fixed point $u^* \in L$ and define $\dot{s} = (\nabla_{e_i} s)|_{u^*}$ and $\dot{\Gamma} = (\nabla_{e_i}\Gamma)|_{u^*}$. Since $\Gamma^2 = -1$, we have $\Gamma\dot{\Gamma} + \dot{\Gamma}\Gamma = 0$, which shows that Γ and $\dot{\Gamma}$ anti-commute. Consequently, $\dot{\Gamma}$ interchanges V_+ and V_-. Differentiating the equation $\Gamma s = \pm j_L s$ yields $\dot{\Gamma}s + \Gamma\dot{s} = \pm j_L\dot{s}$, which implies $(\Gamma \mp j_L)\dot{s} = -\dot{\Gamma}s \in V_\mp$. Restricted to $V_\mp$, the map

$\Gamma \mp j_L : V_\mp \to V_\mp$ acts as $\mp 2j_L$, so its inverse is given by $(\Gamma \mp j_L)^{-1} = (\mp 2j_L)^{-1} = \pm\frac{1}{2}j_L$. Thus,

$$\dot{s} = (\Gamma \mp j_L)^{-1}(\Gamma \mp j_L)\dot{s} = \mp\frac{1}{2}j_L\dot{\Gamma}s.$$

Since $u^* \in L$ was arbitrary, this completes the proof. □

As in the previous subsection, L can be viewed as a complex 1-dimensional Kähler manifold. Additionally, we equip $V_\pm$ with the complex structure $J_\pm = -\Gamma$. That is, $J_+s_1 = -s_2, J_+s_2 = s_1$ on V_+ and $J_-s_3 = -s_4, J_-s_4 = s_3$ on V_-. This turns both V_+ and V_- into rank 1 complex vector bundles over L.

The Levi-Civita connection on S^4 induces connections ∇ on $\$_-(S^4)|_L$ and $\nabla^{V_\pm} = \pi_{V_\pm} \circ \nabla$ on $V_\pm$, where $\pi_{V_\pm} : V_+ \oplus V_- \to V_\pm$ denotes the projection onto $V_\pm$. Let s be a local section of $V_\pm$ and $i \in \{1, 2\}$. According to Lemma 4.6, $\nabla_{e_i}s$ is orthogonal to $V_\pm$, which implies that $\nabla^{V_\pm}_{e_i}s = \pi_{V_\pm}(\nabla_{e_i}s) = 0$. From this, we obtain $(\nabla^{V_\pm}_{e_i}J_\pm)(s) = \nabla^{V_\pm}_{e_i}(J_\pm s) - J_\pm(\nabla^{V_\pm}_{e_i}s) = 0$, showing that $J_\pm$ is parallel with respect to $\nabla^{V_\pm}$. Thus, the connection $\nabla^{V_\pm}$ is complex linear with respect to $J_\pm$. Following the same reasoning as in the previous subsection, we conclude that $V_\pm$ is a holomorphic vector bundle with holomorphic structure given by $\bar{\partial}_{V_\pm} = (\nabla^{V_\pm})^{0,1}$.

Theorem 4.7. *The submanifold $V_+ + \psi$ is Cayley in $\$_-(S^4)$ if and only if L^2 is minimal in S^4 and $\psi \in \Gamma(V_-)$ is holomorphic.*

Remark 4.8. Note that the conditions on L and ψ are the same as in the case of $\mathbb{R}^4$ in [KL12] (see Theorem 3.14).

Proof. The immersion of $V_+ + \psi$ into $M^8 = \$_-(S^4)$, $\Psi : X_\psi \to M$, is locally given by

$$\begin{aligned}\Psi : (u, t) \mapsto &\big(x(u), t_1s_1(u) + t_2s_2(u) + \psi(u)\big)\\ &= \big(x(u), t_1s_1(u) + t_2s_2(u) + a(u)s_3(u) + b(u)s_4(u)\big),\end{aligned}$$

where $u = (u_1, u_2)$ and $t = (t_1, t_2)$ are the coordinates on L and on the fiber, respectively, $x = (x_1, \dots, x_4)$ is the local immersion of L into S^4, and a and b are the coordinates of ψ with respect to the local trivialization $V_- \overset{\text{loc}}{=} \operatorname{span}\{s_3, s_4\}$. Then the tangent space to X_ψ at some fixed point $s^* = \Psi(u^*, t^*) \in M$ is spanned by $E_i = \Psi_*(\partial_{u_i})$ and $F_j = \Psi_*(\partial_{t_j})$ for $i, j = 1, 2$, omitting the dependence on (u^*, t^*).

By Lemma 4.6, we have $\nabla_{e_i}s_k = -\frac{1}{2}j_L(\nabla_{e_i}\Gamma)s_k$ and $\nabla_{e_i}s_l = \frac{1}{2}j_L(\nabla_{e_i}\Gamma)s_l$ for $k = 1, 2$ and $l = 3, 4$. As we are working with normal coordinates at u^*, we can use (3.2) and (3.10) to compute

$$\begin{aligned}\nabla_{e_i}\Gamma &= \gamma(\nabla_{e_i}e^1)\gamma(e^2) + \gamma(e^1)\gamma(\nabla_{e_i}e^2)\\ &= \gamma(-A^3_{i1}\nu^3 - A^4_{i1}\nu^4)\gamma(e^2) + \gamma(e^1)\gamma(-A^3_{i2}\nu^3 - A^4_{i2}\nu^4)\\ &= \frac{1}{2}\big(-A^3_{i1}\gamma(\nu^3 \wedge e^2) - A^4_{i1}\gamma(\nu^4 \wedge e^2) - A^3_{i2}\gamma(e^1 \wedge \nu^3) - A^4_{i2}\gamma(e^1 \wedge \nu^4)\big)\\ &= \frac{1}{4}\big((-A^3_{i2} - A^4_{i1})\gamma(f^2) + (A^3_{i1} - A^4_{i2})\gamma(f^3)\big)\\ &= B_i\frac{1}{4}\gamma(f^2) + C_i\frac{1}{4}\gamma(f^3),\end{aligned}$$

where we set $B_i = -A^3_{i2} - A^4_{i1}$ and $C_i = A^3_{i1} - A^4_{i2}$. Substituting the expressions $s_2 = \frac{1}{4}\gamma(f^1)s_1$, $s_3 = \frac{1}{4}\gamma(f^2)s_1$ and $s_4 = \frac{1}{4}\gamma(f^3)s_1$ into $(\nabla_{e_i}\Gamma)s_k$ and then applying (3.11) yields

$$\begin{aligned}
(\nabla_{e_i}\Gamma)s_1 &= B_i\frac{1}{4}\gamma(f^2)s_1 + C_i\frac{1}{4}\gamma(f^3)s_1 = B_i s_3 + C_i s_4,\\
(\nabla_{e_i}\Gamma)s_2 &= B_i\frac{1}{16}\gamma(f^2)\gamma(f^1)s_1 + C_i\frac{1}{16}\gamma(f^3)\gamma(f^1)s_1 = -B_i\frac{1}{4}\gamma(f^3)s_1 + C_i\frac{1}{4}\gamma(f^2)s_1\\
&= C_i s_3 - B_i s_4,\\
(\nabla_{e_i}\Gamma)s_3 &= B_i\frac{1}{16}\gamma(f^2)\gamma(f^2)s_1 + C_i\frac{1}{16}\gamma(f^3)\gamma(f^2)s_1 = -B_i s_1 - C_i\frac{1}{4}\gamma(f^1)s_1\\
&= -B_i s_1 - C_i s_2,\\
(\nabla_{e_i}\Gamma)s_4 &= B_i\frac{1}{16}\gamma(f^2)\gamma(f^3)s_1 + C_i\frac{1}{16}\gamma(f^3)\gamma(f^3)s_1 = B_i\frac{1}{4}\gamma(f^1)s_1 - C_i s_1\\
&= -C_i s_1 + B_i s_2.
\end{aligned}$$

Using these formulas, we find that E_i takes the form

$$\begin{aligned}
E_i &= \bar{e}_i + \mathrm{Vert}_{s^*}\big(t_1^*\nabla_{e_i}s_1 + t_2^*\nabla_{e_i}s_2 + \nabla_{e_i}(as_3 + bs_4)\big)\\
&= \bar{e}_i + \mathrm{Vert}_{s^*}(a_i s_3 + b_i s_4) + \mathrm{Vert}_{s^*}(t_1^*\nabla_{e_i}s_1 + t_2^*\nabla_{e_i}s_2 + a\nabla_{e_i}s_3 + b\nabla_{e_i}s_4)\\
&= \bar{e}_i + a_i\check{s}_3 + b_i\check{s}_4\\
&\quad - \frac{1}{2}j_L\mathrm{Vert}_{s^*}\big(t_1^*(\nabla_{e_i}\Gamma)s_1 + t_2^*(\nabla_{e_i}\Gamma)s_2\big) + \frac{1}{2}j_L\mathrm{Vert}_{s^*}\big(a(\nabla_{e_i}\Gamma)s_3 + b(\nabla_{e_i}\Gamma)s_4\big)\\
&= \bar{e}_i + a_i\check{s}_3 + b_i\check{s}_4\\
&\quad - \frac{1}{2}j_L\mathrm{Vert}_{s^*}\big((t_1^*B_i + t_2^*C_i)s_3 + (t_1^*C_i - t_2^*B_i)s_4\big)\\
&\quad + \frac{1}{2}j_L\mathrm{Vert}_{s^*}\big((-aB_i - bC_i)s_1 + (-aC_i + bB_i)s_2\big)\\
&= \bar{e}_i + a_i\check{s}_3 + b_i\check{s}_4\\
&\quad - \frac{1}{2}\mathrm{Vert}_{s^*}\big((t_1^*C_i - t_2^*B_i)s_3 - (t_1^*B_i + t_2^*C_i)s_4\big)\\
&\quad + \frac{1}{2}\mathrm{Vert}_{s^*}\big(-(-aC_i + bB_i)s_1 + (-aB_i - bC_i)s_2\big)\\
&= \bar{e}_i + a_i\check{s}_3 + b_i\check{s}_4\\
&\quad + \frac{1}{2}\big((aC_i - bB_i)\check{s}_1 + (-aB_i - bC_i)\check{s}_2 + (-t_1^*C_i + t_2^*B_i)\check{s}_3 + (t_1^*B_i + t_2^*C_i)\check{s}_4\big)
\end{aligned}$$

with $B_i = -A^3_{i2} - A^4_{i1}$, $C_i = A^3_{i1} - A^4_{i2}$ and $a_i = \frac{\partial a}{\partial u_i}$, $b_i = \frac{\partial b}{\partial u_i}$ for $i = 1, 2$. The other two basis vectors are given by

$$F_j = \Psi_*\left(\frac{\partial}{\partial t_j}\right) = \check{s}_j, \qquad j = 1, 2.$$

By Proposition 2.27, X_ψ is Cayley in M^8 if and only if η, defined as

$$\begin{aligned}
\eta(u,v,w,y) &= u^\flat \wedge X(v,w,y)^\flat + v^\flat \wedge X(w,u,y)^\flat + w^\flat \wedge X(u,v,y)^\flat + y^\flat \wedge X(v,u,w)^\flat\\
&\quad + u \lrcorner X(v,w,y) \lrcorner \Phi + v \lrcorner X(w,u,y) \lrcorner \Phi + w \lrcorner X(u,v,y) \lrcorner \Phi + y \lrcorner X(v,u,w) \lrcorner \Phi
\end{aligned}$$

for $u, v, w, y \in T_sM$, $s \in M$, vanishes on X_ψ. Using the formulas

$$\begin{aligned}\Phi &= u^4\bar{e}^1\bar{e}^2\bar{\nu}^3\bar{\nu}^4 - u^2v^2(\omega_1\sigma^1 + \omega_2\sigma^2 + \omega_3\sigma^3) + v^4\check{s}^1\check{s}^2\check{s}^3\check{s}^4 \\ &= u^4\bar{e}^1\bar{e}^2\bar{\nu}^3\bar{\nu}^4 - u^2v^2(\bar{e}^1\bar{e}^2 + \bar{\nu}^3\bar{\nu}^4)(\check{s}^1\check{s}^2 + \check{s}^3\check{s}^4) - u^2v^2(\bar{e}^1\bar{\nu}^3 + \bar{\nu}^4\bar{e}^2)(\check{s}^1\check{s}^3 + \check{s}^4\check{s}^2) \\ &\quad - u^2v^2(\bar{e}^1\bar{\nu}^4 + \bar{e}^2\bar{\nu}^3)(\check{s}^1\check{s}^4 + \check{s}^2\check{s}^3) + v^4\check{s}^1\check{s}^2\check{s}^3\check{s}^4\end{aligned}$$

(see (3.12)) and $X(u, v, w)^\flat = w \lrcorner v \lrcorner u \lrcorner \Phi$ (2.16), we compute

$$\begin{aligned}&E_1^\flat \wedge X(E_2, F_1, F_2)^\flat + E_2^\flat \wedge X(F_1, E_1, F_2)^\flat + F_1^\flat \wedge X(E_1, E_2, F_2)^\flat + F_2^\flat \wedge X(E_2, E_1, F_1)^\flat \\ &\quad = u^2v^4\big((a_1 + b_2)(-\bar{e}^1\check{s}^3 - \bar{e}^2\check{s}^4 + \bar{\nu}^3\check{s}^1 + \bar{\nu}^4\check{s}^2) + (a_2 - b_1)(\bar{e}^1\check{s}^4 - \bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\big) \\ &\qquad + \frac{1}{2}u^2v^4\Big(\big(t_1^*(-B_2 + C_1) + t_2^*(-B_1 - C_2)\big)(\bar{e}^1\check{s}^3 + \bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) \\ &\qquad\qquad + \big(t_1^*(-B_1 - C_2) + t_2^*(B_2 - C_1)\big)(\bar{e}^1\check{s}^4 - \bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\Big)\end{aligned}$$

and

$$\begin{aligned}&E_1 \lrcorner X(E_2, F_1, F_2) \lrcorner \Phi + E_2 \lrcorner X(F_1, E_1, F_2) \lrcorner \Phi + F_1 \lrcorner X(E_1, E_2, F_2) \lrcorner \Phi + F_2 \lrcorner X(E_2, E_1, F_1) \lrcorner \Phi \\ &\quad = 3u^2v^4\big((a_1 + b_2)(\bar{e}^1\check{s}^3 + \bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) + (a_2 - b_1)(-\bar{e}^1\check{s}^4 + \bar{e}^2\check{s}^3 - \bar{\nu}^3\check{s}^2 + \bar{\nu}^4\check{s}^1)\big) \\ &\qquad + \frac{3}{2}u^2v^4\Big(\big(t_1^*(B_2 - C_1) + t_2^*(B_1 + C_2)\big)(\bar{e}^1\check{s}^3 + \bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) \\ &\qquad\qquad + \big(t_1^*(B_1 + C_2) + t_2^*(-B_2 + C_1)\big)(\bar{e}^1\check{s}^4 - \bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\Big),\end{aligned}$$

which add up to

$$\begin{aligned}&\eta(E_1, E_2, F_1, F_2) \\ &\quad = 2u^2v^4\big((a_1 + b_2)(\bar{e}^1\check{s}^3 + \bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) + (a_2 - b_1)(-\bar{e}^1\check{s}^4 + \bar{e}^2\check{s}^3 - \bar{\nu}^3\check{s}^2 + \bar{\nu}^4\check{s}^1)\big) \\ &\qquad + u^2v^4\Big(\big(t_1^*(B_2 - C_1) + t_2^*(B_1 + C_2)\big)(\bar{e}^1\check{s}^3 + \bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) \\ &\qquad\qquad + \big(t_1^*(B_1 + C_2) + t_2^*(-B_2 + C_1)\big)(\bar{e}^1\check{s}^4 - \bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\Big)\end{aligned}$$

(see Appendix A for the complete computation). Given that $(u^*, t^*) \in X_\psi$ was arbitrary, X_ψ is Cayley in M if and only if this expression vanishes on all of X_ψ. Since $u, v > 0$, this is equivalent to the conditions

(I) $a_1 + b_2 = 0$ and $a_2 - b_1 = 0$,

(II) $B_2 - C_1 = 0$ and $B_1 + C_2 = 0$.

Substituting the definitions of B_i and C_i yields

$$\begin{aligned}B_2 - C_1 &= (-A_{22}^3 - A_{12}^4) - (A_{11}^3 - A_{12}^4) = -(A_{11}^3 + A_{22}^3) = -\operatorname{Tr} A^3, \\ B_1 + C_2 &= (-A_{12}^3 - A_{11}^4) + (A_{12}^3 - A_{22}^4) - (A_{11}^4 + A_{22}^4) = -\operatorname{Tr} A^4,\end{aligned}$$

which shows that (II) is equivalent to L being minimal in S^4.

It remains to prove that the first condition is fulfilled if and only if $\psi \in \Gamma(V_-)$ is holomorphic, meaning that $\bar{\partial}_{V_-}\psi = (\nabla^{V_-})^{0,1}\psi = 0$. Since $e_1 + ie_2$ locally trivializes

$(TL)^{0,1}$, the latter is equivalent to $\nabla^{V_-}_{e_1+ie_2}\psi = 0$. Using the formula $\psi = as_3 + bs_4$, we compute

$$\begin{aligned}
\nabla^{V_-}_{e_1+ie_2}\psi &= \nabla^{V_-}_{e_1}\psi + J_-(\nabla^{V_-}_{e_2}\psi) \\
&= \pi_{V_-}(\nabla_{e_1}\psi) + J_-\big(\pi_{V_-}(\nabla_{e_2}\psi)\big) \\
&= \pi_{V_-}(a\nabla_{e_1}s_3 + b\nabla_{e_1}s_4 + a_1s_3 + b_1s_4) \\
&\quad + J_-\big(\pi_{V_-}(a\nabla_{e_2}s_3 + b\nabla_{e_2}s_4 + a_2s_3 + b_2s_4)\big) \\
&= a_1s_3 + b_1s_4 + J_-(a_2s_3 + b_2s_4) \\
&= (a_1 + b_2)s_3 + (-a_2 + b_1)s_4,
\end{aligned}$$

where the fourth equality follows from Lemma 4.6 and the fact that s_1 and s_2 are orthogonal to V_-. Consequently, ψ is holomorphic if and only if (I) holds, which completes the proof. □

Remark 4.9. In the same way, we obtain an analogous result for $\chi + V_-$ with $\chi \in \Gamma(V_+)$.

5. Conclusion

Our findings demonstrate that the constructions of calibrated submanifolds in Euclidean spaces in [KL12] cannot be entirely extended to the manifolds T^*S^n, $\Lambda^2_-(T^*X)$ ($X^4 = S^4, \mathbb{CP}^2$) and $\$_-(S^4)$ considered in [KM05]. While the results for the two spaces of exceptional holonomy are in line with the previous findings, the construction in T^*S^n does not provide any new examples because the Lagrangian condition already requires the 1-form to vanish. As in [KL12], the (co-)associative and Cayley subbundles constructed in [KM05] allow deformations destroying the linear structure of the fiber, while the base space L^2 remains of the same type after twisting, namely minimal or negative superminimal. This implies that the moduli space of calibrated submanifolds near a calibrated subbundle of this kind not only contains deformations of the base L but also of the fiber. In contrast, the special Lagrangian bundle construction in T^*S^n is much more rigid than in the case of $T^*\mathbb{R}^n$.

It would also be interesting to study whether there exist other types of deformations in the above three cases and if we can find similar results for other manifolds of special holonomy.

A. Computation of $\eta(E_1, E_2, F_1, F_2)$

This section provides the calculation of $\eta(E_1, E_2, F_1, F_2)$, which was omitted in Subsection 4.3. Recall that η is defined as

$$\begin{aligned}\eta(u,v,w,y) &= u^\flat \wedge X(v,w,y)^\flat + v^\flat \wedge X(w,u,y)^\flat + w^\flat \wedge X(u,v,y)^\flat + y^\flat \wedge X(v,u,w)^\flat \\ &\quad + u \lrcorner X(v,w,y) \lrcorner \Phi + v \lrcorner X(w,u,y) \lrcorner \Phi + w \lrcorner X(u,v,y) \lrcorner \Phi + y \lrcorner X(v,u,w) \lrcorner \Phi\end{aligned}$$

for $u, v, w, y \in T_sM$, $s \in M$ (see Proposition 2.27), where $X(u,v,w)^\flat = w \lrcorner v \lrcorner u \lrcorner \Phi$ (2.16). By (3.12), Φ restricted to L is locally given by

$$\begin{aligned}\Phi &= u^4 \bar{e}^1 \bar{e}^2 \bar{\nu}^3 \bar{\nu}^4 - u^2 v^2 (\omega_1 \sigma^1 + \omega_2 \sigma^2 + \omega_3 \sigma^3) + v^4 \check{s}^1 \check{s}^2 \check{s}^3 \check{s}^4 \\ &= u^4 \bar{e}^1 \bar{e}^2 \bar{\nu}^3 \bar{\nu}^4 - u^2 v^2 (\bar{e}^1 \bar{e}^2 + \bar{\nu}^3 \bar{\nu}^4)(\check{s}^1 \check{s}^2 + \check{s}^3 \check{s}^4) - u^2 v^2 (\bar{e}^1 \bar{\nu}^3 + \bar{\nu}^4 \bar{e}^2)(\check{s}^1 \check{s}^3 + \check{s}^4 \check{s}^2) \\ &\quad - u^2 v^2 (\bar{e}^1 \bar{\nu}^4 + \bar{e}^2 \bar{\nu}^3)(\check{s}^1 \check{s}^4 + \check{s}^2 \check{s}^3) + v^4 \check{s}^1 \check{s}^2 \check{s}^3 \check{s}^4.\end{aligned}$$

In Subsection 4.3, we then defined the tangent vectors

$$\begin{aligned}E_i &= \bar{e}_i + a_i \check{s}_3 + b_i \check{s}_4 \\ &\quad + \frac{1}{2}\left((aC_i - bB_i)\check{s}_1 + (-aB_i - bC_i)\check{s}_2 + (-t_1 C_i + t_2 B_i)\check{s}_3 + (t_1 B_i + t_2 C_i)\check{s}_4\right), \\ F_j &= \check{s}_j\end{aligned}$$

for $i, j = 1, 2$, where $B_i = -A^3_{i2} - A^4_{i1}$, $C_i = A^3_{i1} - A^4_{i2}$ and $a_i = \frac{\partial a}{\partial u_i}$, $b_i = \frac{\partial b}{\partial u_i}$.

We start by computing the interior product of Φ with each of these vectors:

$$F_1 \lrcorner \Phi = -u^2 v^2 (\omega_1 \check{s}^2 + \omega_2 \check{s}^3 + \omega_3 \check{s}^4) + v^4 \check{s}^2 \check{s}^3 \check{s}^4,$$

$$F_2 \lrcorner \Phi = -u^2 v^2 (-\omega_1 \check{s}^1 - \omega_2 \check{s}^4 + \omega_3 \check{s}^3) - v^4 \check{s}^1 \check{s}^3 \check{s}^4,$$

$$\begin{aligned}E_1 \lrcorner \Phi &= u^4 \bar{e}^2 \bar{\nu}^3 \bar{\nu}^4 - u^2 v^2 (\bar{e}^2 \sigma^1 + \bar{\nu}^3 \sigma^2 + \bar{\nu}^4 \sigma^3) \\ &\quad - u^2 v^2 a_1 (\omega_1 \check{s}^4 - \omega_2 \check{s}^1 - \omega_3 \check{s}^2) + v^4 a_1 \check{s}^1 \check{s}^2 \check{s}^4 \\ &\quad - u^2 v^2 b_1 (-\omega_1 \check{s}^3 + \omega_2 \check{s}^2 - \omega_3 \check{s}^1) - v^4 b_1 \check{s}^1 \check{s}^2 \check{s}^3 \\ &\quad - \frac{1}{2} u^2 v^2 \big((aC_1 - bB_1)(\omega_1 \check{s}^2 + \omega_2 \check{s}^3 + \omega_3 \check{s}^4) \\ &\qquad + (-aB_1 - bC_1)(-\omega_1 \check{s}^1 - \omega_2 \check{s}^4 + \omega_3 \check{s}^3) \\ &\qquad + (-t_1 C_1 + t_2 B_1)(\omega_1 \check{s}^4 - \omega_2 \check{s}^1 - \omega_3 \check{s}^2) \\ &\qquad + (t_1 B_1 + t_2 C_1)(-\omega_1 \check{s}^3 + \omega_2 \check{s}^2 - \omega_3 \check{s}^1)\big) \\ &\quad + \frac{1}{2} v^4 \big((aC_1 - bB_1)\check{s}^2 \check{s}^3 \check{s}^4 - (-aB_1 - bC_1)\check{s}^1 \check{s}^3 \check{s}^4 \\ &\qquad + (-t_1 C_1 + t_2 B_1)\check{s}^1 \check{s}^2 \check{s}^4 - (t_1 B_1 + t_2 C_1)\check{s}^1 \check{s}^2 \check{s}^3\big),\end{aligned}$$

$$\begin{aligned}E_2 \lrcorner \Phi &= -u^4 \bar{e}^1 \bar{\nu}^3 \bar{\nu}^4 - u^2 v^2 (-\bar{e}^1 \sigma^1 - \bar{\nu}^4 \sigma^2 + \bar{\nu}^3 \sigma^3) \\ &\quad - u^2 v^2 a_2 (\omega_1 \check{s}^4 - \omega_2 \check{s}^1 - \omega_3 \check{s}^2) + v^4 a_2 \check{s}^1 \check{s}^2 \check{s}^4 \\ &\quad - u^2 v^2 b_2 (-\omega_1 \check{s}^3 + \omega_2 \check{s}^2 - \omega_3 \check{s}^1) - v^4 b_2 \check{s}^1 \check{s}^2 \check{s}^3\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}u^2v^2\big((aC_2 - bB_2)(\omega_1\check{s}^2 + \omega_2\check{s}^3 + \omega_3\check{s}^4) \\
&\qquad + (-aB_2 - bC_2)(-\omega_1\check{s}^1 - \omega_2\check{s}^4 + \omega_3\check{s}^3) \\
&\qquad + (-t_1C_2 + t_2B_2)(\omega_1\check{s}^4 - \omega_2\check{s}^1 - \omega_3\check{s}^2) \\
&\qquad + (t_1B_2 + t_2C_2)(-\omega_1\check{s}^3 + \omega_2\check{s}^2 - \omega_3\check{s}^1)\big) \\
&+ \frac{1}{2}v^4\big((aC_2 - bB_2)\check{s}^2\check{s}^3\check{s}^4 - (-aB_2 - bC_2)\check{s}^1\check{s}^3\check{s}^4 \\
&\qquad + (-t_1C_2 + t_2B_2)\check{s}^1\check{s}^2\check{s}^4 - (t_1B_2 + t_2C_2)\check{s}^1\check{s}^2\check{s}^3\big).
\end{aligned}
$$

From this, we deduce

$$
\begin{aligned}
F_2\lrcorner F_1\lrcorner\Phi &= -u^2v^2\omega_1 + v^4\check{s}^3\check{s}^4, \\
E_1\lrcorner F_2\lrcorner F_1\lrcorner\Phi &= -u^2v^2\bar{e}^2 + v^4(a_1\check{s}^4 - b_1\check{s}^3) + \frac{1}{2}v^4\big((-t_1C_1 + t_2B_1)\check{s}^4 - (t_1B_1 + t_2C_1)\check{s}^3\big), \\
E_2\lrcorner F_2\lrcorner F_1\lrcorner\Phi &= u^2v^2\bar{e}^1 + v^4(a_2\check{s}^4 - b_2\check{s}^3) + \frac{1}{2}v^4\big((-t_1C_2 + t_2B_2)\check{s}^4 - (t_1B_2 + t_2C_2)\check{s}^3\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&E_2\lrcorner E_1\lrcorner\Phi \\
&\quad = u^4\bar{\nu}^3\bar{\nu}^4 + v^4(a_1b_2 - a_2b_1)\check{s}^1\check{s}^2 + u^2v^2(a_2b_1 - a_1b_2)\omega_1 - u^2v^2\sigma^1 \\
&\qquad + u^2v^2\big(-a_1(-\bar{e}^1\check{s}^4 + \bar{\nu}^4\check{s}^1 - \bar{\nu}^3\check{s}^2) + a_2(\bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) \\
&\qquad\qquad - b_1(\bar{e}^1\check{s}^3 - \bar{\nu}^4\check{s}^2 - \bar{\nu}^3\check{s}^1) + b_2(-\bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\big) \\
&\qquad + \frac{1}{2}u^2v^2\,\big((aC_1 - bB_1)(-a_2\omega_2 - b_2\omega_3) + (-aB_1 - bC_1)(b_2\omega_2 - a_2\omega_3) \\
&\qquad\qquad + (aC_2 - bB_2)(a_1\omega_2 + b_1\omega_3) + (-aB_2 - bC_2)(-b_1\omega_2 + a_1\omega_3)\big) \\
&\qquad + \frac{1}{2}u^2v^2\big(-a_1(t_1B_2 + t_2C_2) + a_2(t_1B_1 + t_2C_1) \\
&\qquad\qquad + b_1(-t_1C_2 + t_2B_2) - b_2(-t_1C_1 + t_2B_1)\big)\omega_1 \\
&\qquad + \frac{1}{2}u^2v^2\big(-(aC_1 - bB_1)(-\bar{e}^1\check{s}^2 - \bar{\nu}^4\check{s}^3 + \bar{\nu}^3\check{s}^4) - (-aB_1 - bC_1)(\bar{e}^1\check{s}^1 + \bar{\nu}^4\check{s}^4 + \bar{\nu}^3\check{s}^3) \\
&\qquad\qquad + (aC_2 - bB_2)(\bar{e}^2\check{s}^2 + \bar{\nu}^3\check{s}^3 + \bar{\nu}^4\check{s}^4) + (-aB_2 - bC_2)(-\bar{e}^2\check{s}^1 - \bar{\nu}^3\check{s}^4 + \bar{\nu}^4\check{s}^3) \\
&\qquad\qquad - (-t_1C_1 + t_2B_1)(-\bar{e}^1\check{s}^4 + \bar{\nu}^4\check{s}^1 - \bar{\nu}^3\check{s}^2) - (t_1B_1 + t_2C_1)(\bar{e}^1\check{s}^3 - \bar{\nu}^4\check{s}^2 - \bar{\nu}^3\check{s}^1) \\
&\qquad\qquad + (-t_1C_2 + t_2B_2)(\bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) + (t_1B_2 + t_2C_2)(-\bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\big) \\
&\qquad - \frac{1}{4}u^2v^2\Big((aC_1 - bB_1)\big((-aB_2 - bC_2)\omega_1 + (-t_1C_2 + t_2B_2)\omega_2 + (t_1B_2 + t_2C_2)\omega_3\big) \\
&\qquad\qquad + (-aB_1 - bC_1)\big(-(aC_2 - bB_2)\omega_1 - (t_1B_2 + t_2C_2)\omega_2 + (-t_1C_2 + t_2B_2)\omega_3\big) \\
&\qquad\qquad + (-t_1C_1 + t_2B_1)\big((t_1B_2 + t_2C_2)\omega_1 - (aC_2 - bB_2)\omega_2 - (-aB_2 - bC_2)\omega_3\big) \\
&\qquad\qquad + (t_1B_1 + t_2C_1)\big(-(-t_1C_2 + t_2B_2)\omega_1 + (-aB_2 - bC_2)\omega_2 - (aC_2 - bB_2)\omega_3\big)\Big) \\
&\qquad + \frac{1}{2}v^4\big((aC_1 - bB_1)(-a_2\check{s}^2\check{s}^4 + b_2\check{s}^2\check{s}^3) - (-aB_1 - bC_1)(-a_2\check{s}^1\check{s}^4 + b_2\check{s}^1\check{s}^3) \\
&\qquad\quad + (aC_2 - bB_2)(a_1\check{s}^2\check{s}^4 - b_1\check{s}^2\check{s}^3) + (-aB_2 - bC_2)(-a_1\check{s}^1\check{s}^4 + b_1\check{s}^1\check{s}^3)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}v^4\big(a_1(t_1B_2+t_2C_2)-a_2(t_1B_1+t_2C_1)\\
&\qquad -b_1(-t_1C_2+t_2B_2)+b_2(-t_1C_1+t_2B_1)\big)\check{s}^1\check{s}^2\\
&+\frac{1}{4}v^4\Big((aC_1-bB_1)\big((-aB_2-bC_2)\check{s}^3\check{s}^4-(-t_1C_2+t_2B_2)\check{s}^2\check{s}^4+(t_1B_2+t_2C_2)\check{s}^2\check{s}^3\big)\\
&\qquad -(-aB_1-bC_1)\big((aC_2-bB_2)\check{s}^3\check{s}^4-(-t_1C_2+t_2B_2)\check{s}^1\check{s}^4+(t_1B_2+t_2C_2)\check{s}^1\check{s}^3\big)\\
&\qquad +(-t_1C_1+t_2B_1)\big((aC_2-bB_2)\check{s}^2\check{s}^4-(-aB_2-bC_2)\check{s}^1\check{s}^4+(t_1B_2+t_2C_2)\check{s}^1\check{s}^2\big)\\
&\qquad -(t_1B_1+t_2C_1)\big((aC_2-bB_2)\check{s}^2\check{s}^3-(-aB_2-bC_2)\check{s}^1\check{s}^3+(-t_1C_2+t_2B_2)\check{s}^1\check{s}^2\big)\Big).
\end{aligned}
$$

The latter implies

$$
\begin{aligned}
&F_1\lrcorner E_2\lrcorner E_1\lrcorner\Phi\\
&\quad = v^4(a_1b_2-a_2b_1)\check{s}^2-u^2v^2\check{s}^2+u^2v^2\big((a_2-b_1)\bar{\nu}^3+(a_1+b_2)\bar{\nu}^4\big)\\
&\qquad +\frac{1}{2}u^2v^2\Big((-aB_1-bC_1)\bar{e}^1+(-aB_2-bC_2)\bar{e}^2\\
&\qquad\qquad +\big(t_1(-B_1-C_2)+t_2(B_2-C_1)\big)\bar{\nu}^3+\big(t_1(B_2-C_1)+t_2(B_2+C_2)\big)\bar{\nu}^4\Big)\\
&\qquad +\frac{1}{2}v^4\big(-(-aB_1-bC_1)(-a_2\check{s}^4+b_2\check{s}^3)+(-aB_2-bC_2)(-a_1\check{s}^4+b_1\check{s}^3)\big)\\
&\qquad +\frac{1}{2}v^4\big(a_1(t_1B_2+t_2C_2)-a_2(t_1B_1+t_2C_1)\\
&\qquad\qquad -b_1(-t_1C_2+t_2B_2)+b_2(-t_1C_1+t_2B_1)\big)\check{s}^2\\
&\qquad +\frac{1}{4}v^4\Big(-(-aB_1-bC_1)\big(-(-t_1C_2+t_2B_2)\check{s}^4+(t_1B_2+t_2C_2)\check{s}^3\big)\\
&\qquad\qquad +(-t_1C_1+t_2B_1)\big(-(-aB_2-bC_2)\check{s}^4+(t_1B_2+t_2C_2)\check{s}^2\big)\\
&\qquad\qquad -(t_1B_1+t_2C_1)\big(-(-aB_2-bC_2)\check{s}^3+(-t_1C_2+t_2B_2)\check{s}^2\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&F_2\lrcorner E_2\lrcorner E_1\lrcorner\Phi\\
&\quad = -v^4(a_1b_2-a_2b_1)\check{s}^1+u^2v^2\check{s}^1+u^2v^2\big((-a_1-b_2)\bar{\nu}^3+(a_2-b_1)\bar{\nu}^4\big)\\
&\qquad +\frac{1}{2}u^2v^2\Big(-(aC_1-bB_1)\bar{e}^1-(aC_2-bB_2)\bar{e}^2\\
&\qquad\qquad +\big(t_1(-B_2+C_1)+t_2(-B_1-C_2)\big)\bar{\nu}^3+\big(t_1(-B_1-C_2)+t_2(B_2-C_1)\big)\bar{\nu}^4\Big)\\
&\qquad +\frac{1}{2}v^4\big((aC_1-bB_1)(-a_2\check{s}^4+b_2\check{s}^3)+(aC_2-bB_2)(a_1\check{s}^4-b_1\check{s}^3)\big)\\
&\qquad -\frac{1}{2}v^4\big(a_1(t_1B_2+t_2C_2)-a_2(t_1B_1+t_2C_1)\\
&\qquad\qquad -b_1(-t_1C_2+t_2B_2)+b_2(-t_1C_1+t_2B_1)\big)\check{s}^1\\
&\qquad +\frac{1}{4}v^4\Big((aC_1-bB_1)\big(-(-t_1C_2+t_2B_2)\check{s}^4+(t_1B_2+t_2C_2)\check{s}^3\big)\\
&\qquad\qquad +(-t_1C_1+t_2B_1)\big((aC_2-bB_2)\check{s}^4-(t_1B_2+t_2C_2)\check{s}^1\big)\\
&\qquad\qquad -(t_1B_1+t_2C_1)\big((aC_2-bB_2)\check{s}^3-(-t_1C_2+t_2B_2)\check{s}^1\big)\Big).
\end{aligned}
$$

By (2.16), we obtain the terms of the form $X(\cdot,\cdot,\cdot)^\flat$ in η from the expressions above via

$$X(E_2,F_1,F_2)^\flat = E_2\lrcorner F_2\lrcorner F_1\lrcorner\Phi, \qquad X(F_1,E_1,F_2)^\flat = -E_1\lrcorner F_2\lrcorner F_1\lrcorner\Phi,$$
$$X(E_1,E_2,F_2)^\flat = F_2\lrcorner E_2\lrcorner E_1\lrcorner\Phi, \qquad X(E_2,E_1,F_1)^\flat = -F_1\lrcorner E_2\lrcorner E_1\lrcorner\Phi.$$

Moreover, we have $\bar{e}_i^\flat = g_M(\cdot,\bar{e}_i) = u^2 g_{\mathcal{H}}(\pi_{\mathcal{H}}\cdot,\bar{e}_i) = u^2\bar{e}^i$, $\bar{\nu}_i^\flat = u^2\bar{\nu}^i$ and $\check{s}_i^\flat = g_M(\cdot,\check{s}_i) = v^2 g_{\mathcal{H}}(\pi_{\mathcal{V}}\cdot,\check{s}_i) = v^2\check{s}^i$, where $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ denote the projections of M onto $\mathcal{H}$ and $\mathcal{V}$, respectively (cf. Theorem 3.17). Thus,

$$\begin{aligned}
E_i^\flat &= u^2\bar{e}^i + v^2(a_i\check{s}^3 + b_i\check{s}^4)\\
&\quad + \frac{1}{2}v^2\big((aC_i - bB_i)\check{s}^1 + (-aB_i - bC_i)\check{s}^2 + (-t_1C_i + t_2B_i)\check{s}^3 + (t_1B_i + t_2C_i)\check{s}^4\big),\\
F_j^\flat &= v^2\check{s}^j
\end{aligned}$$

for $i,j = 1,2$. By combining these formulas, we can calculate the first four terms of η:

$$\begin{aligned}
&E_1^\flat \wedge X(E_2,F_1,F_2)^\flat\\
&\quad = v^6(a_1a_2 + b_1b_2)\check{s}^3\check{s}^4 + u^2v^4\big((-a_1 - b_2)\bar{e}^1\check{s}^3 + (a_2 - b_1)\bar{e}^1\check{s}^4\big)\\
&\qquad + \frac{1}{2}u^2v^4\Big(-(aC_1 - bB_1)\bar{e}^1\check{s}^1 - (-aB_1 - bC_1)\bar{e}^1\check{s}^2\\
&\qquad\qquad + \big(t_1(-B_2 + C_1) + t_2(-B_1 - C_2)\big)\bar{e}^1\check{s}^3\\
&\qquad\qquad\quad + \big(t_1(-B_1 - C_2) + t_2(B_2 - C_1)\big)\bar{e}^1\check{s}^4\Big)\\
&\qquad + \frac{1}{2}v^6\big((aC_1 - bB_1)(a_2\check{s}^1\check{s}^4 - b_2\check{s}^1\check{s}^3) + (-aB_1 - bC_1)(a_2\check{s}^2\check{s}^4 - b_2\check{s}^2\check{s}^3)\big)\\
&\qquad + \frac{1}{2}v^6\big(a_1(-t_1C_2 + t_2B_2) + a_2(-t_1C_1 + t_2B_1)\\
&\qquad\qquad + b_1(t_1B_2 + t_2C_2) + b_2(t_1B_1 + t_2C_1)\big)\check{s}^3\check{s}^4\\
&\qquad + \frac{1}{4}v^6\Big((aC_1 - bB_1)\big((-t_1C_2 + t_2B_2)\check{s}^1\check{s}^4 - (t_1B_2 + t_2C_2)\check{s}^1\check{s}^3\big)\\
&\qquad\qquad + (-aB_1 - bC_1)\big((-t_1C_2 + t_2B_2)\check{s}^2\check{s}^4 - (t_1B_2 + t_2C_2)\check{s}^2\check{s}^3\big)\\
&\qquad\qquad + \big((-t_1C_1 + t_2B_1)(-t_1C_2 + t_2B_2) + (t_1B_1 + t_2C_1)(t_1B_2 + t_2C_2)\big)\check{s}^3\check{s}^4\Big),
\end{aligned}$$

$$\begin{aligned}
&E_2^\flat \wedge X(F_1,E_1,F_2)^\flat\\
&\quad = -v^6(a_1a_2 + b_1b_2)\check{s}^3\check{s}^4 + u^2v^4\big((-a_2 + b_1)\bar{e}^2\check{s}^3 + (-a_1 - b_2)\bar{e}^2\check{s}^4\big)\\
&\qquad + \frac{1}{2}u^2v^4\Big(-(aC_2 - bB_2)\bar{e}^2\check{s}^1 - (-aB_2 - bC_2)\bar{e}^2\check{s}^2\\
&\qquad\qquad + \big(t_1(B_1 + C_2) + t_2(-B_2 + C_1)\big)\bar{e}^2\check{s}^3\\
&\qquad\qquad\quad + \big(t_1(-B_2 + C_1) + t_2(-B_1 - C_2)\big)\bar{e}^2\check{s}^4\Big)\\
&\qquad - \frac{1}{2}v^6\big((aC_2 - bB_2)(a_1\check{s}^1\check{s}^4 - b_1\check{s}^1\check{s}^3) + (-aB_2 - bC_2)(a_1\check{s}^2\check{s}^4 - b_1\check{s}^2\check{s}^3)\big)\\
&\qquad - \frac{1}{2}v^6\big(a_1(-t_1C_2 + t_2B_2) + a_2(-t_1C_1 + t_2B_1)\\
&\qquad\qquad + b_1(t_1B_2 + t_2C_2) + b_2(t_1B_1 + t_2C_1)\big)\check{s}^3\check{s}^4
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}v^6\Big((aC_2-bB_2)\big((-t_1C_1+t_2B_1)\check{s}^1\check{s}^4-(t_1B_1+t_2C_1)\check{s}^1\check{s}^3\big)\\
&\qquad+(-aB_2-bC_2)\big((-t_1C_1+t_2B_1)\check{s}^2\check{s}^4-(t_1B_1+t_2C_1)\check{s}^2\check{s}^3\big)\\
&\qquad+\big((-t_1C_2+t_2B_2)(-t_1C_1+t_2B_1)+(t_1B_2+t_2C_2)(t_1B_1+t_2C_1)\big)\check{s}^3\check{s}^4\Big),
\end{aligned}
$$

$$
\begin{aligned}
&F_1^\flat\wedge X(E_1,E_2,F_2)^\flat\\
&\quad=-u^2v^4\big((-a_1-b_2)\bar{\nu}^3\check{s}^1+(a_2-b_1)\bar{\nu}^4\check{s}^1\big)\\
&\quad-\frac{1}{2}u^2v^4\Big(-(aC_1-bB_1)\bar{e}^1\check{s}^1-(aC_2-bB_2)\bar{e}^2\check{s}^1\\
&\qquad\quad+\big(t_1(-B_2+C_1)+t_2(-B_1-C_2)\big)\bar{\nu}^3\check{s}^1\\
&\qquad\quad\;+\big(t_1(-B_1-C_2)+t_2(B_2-C_1)\big)\bar{\nu}^4\check{s}^1\Big)\\
&\quad+\frac{1}{2}v^6\big((aC_1-bB_1)(-a_2\check{s}^1\check{s}^4+b_2\check{s}^1\check{s}^3)+(aC_2-bB_2)(a_1\check{s}^1\check{s}^4-b_1\check{s}^1\check{s}^3)\big)\\
&\quad+\frac{1}{4}v^6\Big((aC_1-bB_1)\big(-(-t_1C_2+t_2B_2)\check{s}^1\check{s}^4+(t_1B_2+t_2C_2)\check{s}^1\check{s}^3\big)\\
&\qquad+(aC_2-bB_2)\big((-t_1C_1+t_2B_1)\check{s}^1\check{s}^4-(t_1B_1+t_2C_1)\check{s}^1\check{s}^3\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&F_2^\flat\wedge X(E_2,E_1,F_1)^\flat\\
&\quad=u^2v^4\big((a_2-b_1)\bar{\nu}^3\check{s}^2+(a_1+b_2)\bar{\nu}^4\check{s}^2\big)\\
&\quad+\frac{1}{2}u^2v^4\Big((-aB_1-bC_1)\bar{e}^1\check{s}^2+(-aB_2-bC_2)\bar{e}^2\check{s}^2\\
&\qquad\quad+\big(t_1(-B_1-C_2)+t_2(B_2-C_1)\big)\bar{\nu}^3\check{s}^2\\
&\qquad\quad\;+\big(t_1(B_2-C_1)+t_2(B_2+C_2)\big)\bar{\nu}^4\check{s}^2\Big)\\
&\quad-\frac{1}{2}v^6\big(-(-aB_1-bC_1)(-a_2\check{s}^2\check{s}^4+b_2\check{s}^2\check{s}^3)+(-aB_2-bC_2)(-a_1\check{s}^2\check{s}^4+b_1\check{s}^2\check{s}^3)\big)\\
&\quad-\frac{1}{4}v^6\Big(-(-aB_1-bC_1)\big(-(-t_1C_2+t_2B_2)\check{s}^2\check{s}^4+(t_1B_2+t_2C_2)\check{s}^2\check{s}^3\big)\\
&\qquad+(-aB_2-bC_2)\big(-(-t_1C_1+t_2B_1)\check{s}^2\check{s}^4+(t_1B_1+t_2C_1)\check{s}^2\check{s}^3\big)\Big).
\end{aligned}
$$

These expressions add up to

$$
\begin{aligned}
&E_1^\flat\wedge X(E_2,F_1,F_2)^\flat+E_2^\flat\wedge X(F_1,E_1,F_2)^\flat+F_1^\flat\wedge X(E_1,E_2,F_2)^\flat+F_2^\flat\wedge X(E_2,E_1,F_1)^\flat\\
&\quad=u^2v^4\big((a_1+b_2)(-\bar{e}^1\check{s}^3-\bar{e}^2\check{s}^4+\bar{\nu}^3\check{s}^1+\bar{\nu}^4\check{s}^2)+(a_2-b_1)(\bar{e}^1\check{s}^4-\bar{e}^2\check{s}^3+\bar{\nu}^3\check{s}^2-\bar{\nu}^4\check{s}^1)\big)\\
&\quad+\frac{1}{2}u^2v^4\Big(\big(t_1(-B_2+C_1)+t_2(-B_1-C_2)\big)(\bar{e}^1\check{s}^3+\bar{e}^2\check{s}^4-\bar{\nu}^3\check{s}^1-\bar{\nu}^4\check{s}^2)\\
&\qquad+\big(t_1(-B_1-C_2)+t_2(B_2-C_1)\big)(\bar{e}^1\check{s}^4-\bar{e}^2\check{s}^3+\bar{\nu}^3\check{s}^2-\bar{\nu}^4\check{s}^1)\Big).
\end{aligned}
$$

To calculate the remaining four terms, we first need to determine the terms of the form $X(\cdot,\cdot,\cdot)$ in η. Using the identities $(\bar{e}^i)^\sharp=u^{-2}\bar{e}_i$, $(\bar{\nu}^i)^\sharp=u^{-2}\bar{\nu}_i$ and $(\check{s}^i)^\sharp=v^{-2}\check{s}_i$, we find

$$
\begin{aligned}
X(E_2,F_1,F_2)&=(E_2\lrcorner F_2\lrcorner F_1\lrcorner\Phi)^\sharp\\
&=v^2\bar{e}_1+v^2(a_2\check{s}_4-b_2\check{s}_3)+\frac{1}{2}v^2\big((-t_1C_2+t_2B_2)\check{s}_4-(t_1B_2+t_2C_2)\check{s}_3\big),
\end{aligned}
$$

$$
\begin{aligned}
-X(F_1,E_1,F_2) &= (E_1 \lrcorner F_2 \lrcorner F_1 \lrcorner \Phi)^\sharp \\
&= -v^2\bar{e}_2 + v^2(a_1\check{s}_4 - b_1\check{s}_3) + \frac{1}{2}v^2\big((-t_1C_1 + t_2B_1)\check{s}_4 - (t_1B_1 + t_2C_1)\check{s}_3\big),
\end{aligned}
$$

and

$$
\begin{aligned}
X(E_1,&E_2,F_2) = (F_2 \lrcorner E_2 \lrcorner E_1 \lrcorner \Phi)^\sharp \\
&= -v^2(a_1b_2 - a_2b_1)\check{s}_1 + u^2\check{s}_1 + v^2\big((-a_1 - b_2)\bar{\nu}_3 + (a_2 - b_1)\bar{\nu}_4\big) \\
&\quad + \frac{1}{2}v^2\Big(-(aC_1 - bB_1)\bar{e}_1 - (aC_2 - bB_2)\bar{e}_2 \\
&\qquad + \big(t_1(-B_2 + C_1) + t_2(-B_1 - C_2)\big)\bar{\nu}_3 + \big(t_1(-B_1 - C_2) + t_2(B_2 - C_1)\big)\bar{\nu}_4\Big) \\
&\quad + \frac{1}{2}v^2\left((aC_1 - bB_1)(-a_2\check{s}_4 + b_2\check{s}_3) + (aC_2 - bB_2)(a_1\check{s}_4 - b_1\check{s}_3)\right) \\
&\quad - \frac{1}{2}v^2\big(a_1(t_1B_2 + t_2C_2) - a_2(t_1B_1 + t_2C_1) \\
&\qquad - b_1(-t_1C_2 + t_2B_2) + b_2(-t_1C_1 + t_2B_1)\big)\check{s}_1 \\
&\quad + \frac{1}{4}v^2\Big((aC_1 - bB_1)\big(-(-t_1C_2 + t_2B_2)\check{s}_4 + (t_1B_2 + t_2C_2)\check{s}_3\big) \\
&\qquad + (-t_1C_1 + t_2B_1)\big((aC_2 - bB_2)\check{s}_4 - (t_1B_2 + t_2C_2)\check{s}_1\big) \\
&\qquad - (t_1B_1 + t_2C_1)\big((aC_2 - bB_2)\check{s}_3 - (-t_1C_2 + t_2B_2)\check{s}_1\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
-\,X(E_2,&E_1,F_1) = (F_1 \lrcorner E_2 \lrcorner E_1 \lrcorner \Phi)^\sharp \\
&= v^2(a_1b_2 - a_2b_1)\check{s}_2 - u^2\check{s}_2 + v^2\left((a_2 - b_1)\bar{\nu}_3 + (a_1 + b_2)\bar{\nu}_4\right) \\
&\quad + \frac{1}{2}v^2\Big((-aB_1 - bC_1)\bar{e}_1 + (-aB_2 - bC_2)\bar{e}_2 \\
&\qquad + \big(t_1(-B_1 - C_2) + t_2(B_2 - C_1)\big)\bar{\nu}_3 + \big(t_1(B_2 - C_1) + t_2(B_2 + C_2)\big)\bar{\nu}_4\Big) \\
&\quad + \frac{1}{2}v^2\left(-(-aB_1 - bC_1)(-a_2\check{s}_4 + b_2\check{s}_3) + (-aB_2 - bC_2)(-a_1\check{s}_4 + b_1\check{s}_3)\right) \\
&\quad + \frac{1}{2}v^2\big(a_1(t_1B_2 + t_2C_2) - a_2(t_1B_1 + t_2C_1) \\
&\qquad - b_1(-t_1C_2 + t_2B_2) + b_2(-t_1C_1 + t_2B_1)\big)\check{s}_2 \\
&\quad + \frac{1}{4}v^2\Big(-(-aB_1 - bC_1)\left(-(-t_1C_2 + t_2B_2)\check{s}_4 + (t_1B_2 + t_2C_2)\check{s}_3\right) \\
&\qquad + (-t_1C_1 + t_2B_1)\left(-(-aB_2 - bC_2)\check{s}_4 + (t_1B_2 + t_2C_2)\check{s}_2\right) \\
&\qquad - (t_1B_1 + t_2C_1)\left(-(-aB_2 - bC_2)\check{s}_3 + (-t_1C_2 + t_2B_2)\check{s}_2\right)\Big).
\end{aligned}
$$

Computing the interior product of the 3-forms $E_i \lrcorner \Phi$ and $F_j \lrcorner \Phi$ (see page 52) with these vectors yields the desired terms:

$$
\begin{aligned}
-\,E_1 \lrcorner X(&E_2,F_1,F_2) \lrcorner \Phi = X(E_2,F_1,F_2) \lrcorner E_1 \lrcorner \Phi \\
&= u^2v^4\big((-a_1 - b_2)(\bar{e}^2\check{s}^4 - \bar{\nu}^3\check{s}^1 - \bar{\nu}^4\check{s}^2) + (a_2 - b_1)(-\bar{e}^2\check{s}^3 + \bar{\nu}^3\check{s}^2 - \bar{\nu}^4\check{s}^1)\big) \\
&\quad - u^2v^4(a_1a_2 + b_1b_2)\omega_1 + v^6(a_1a_2 + b_1b_2)\check{s}^1\check{s}^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}u^2v^4\big(a_1(-t_1C_2+t_2B_2)+a_2(-t_1C_1+t_2B_1)\\
&\qquad\qquad+b_1(t_1B_2+t_2C_2)+b_2(t_1B_1+t_2C_1)\big)\omega_1\\
&-\frac{1}{2}u^2v^4\left((aC_1-bB_1)(-b_2\omega_2+a_2\omega_3)+(-aB_1-bC_1)(-a_2\omega_2-b_2\omega_3)\right)\\
&-\frac{1}{2}u^2v^4\Big((aC_1-bB_1)(\bar e^2\check s^2+\bar\nu^3\check s^3+\bar\nu^4\check s^4)+(-aB_1-bC_1)(-\bar e^2\check s^1-\bar\nu^3\check s^4+\bar\nu^4\check s^3)\\
&\qquad+\big(t_1(B_2-C_1)+t_2(B_1+C_2)\big)(\bar e^2\check s^4-\bar\nu^3\check s^1-\bar\nu^4\check s^2)\\
&\qquad\qquad\qquad+\big(t_1(B_1+C_2)+t_2(-B_2+C_1)\big)(-\bar e^2\check s^3+\bar\nu^3\check s^2-\bar\nu^4\check s^1)\Big)\\
&-\frac{1}{4}u^2v^4\Big((aC_1-bB_1)\big(-(t_1B_2+t_2C_2)\omega_2+(-t_1C_2+t_2B_2)\omega_3\big)\\
&\qquad+(-aB_1-bC_1)\big(-(-t_1C_2+t_2B_2)\omega_2-(t_1B_2+t_2C_2)\omega_3\big)\\
&\qquad+\big((-t_1C_1+t_2B_1)(-t_1C_2+t_2B_2)+(t_1B_1+t_2C_1)(t_1B_2+t_2C_2)\big)\omega_1\Big)\\
&+\frac{1}{2}v^6\big((aC_1-bB_1)(a_2\check s^2\check s^3+b_2\check s^2\check s^4)-(-aB_1-bC_1)(a_2\check s^1\check s^3+b_2\check s^1\check s^4)\big)\\
&+\frac{1}{2}v^6\big(a_1(-t_1C_2+t_2B_2)+a_2(-t_1C_1+t_2B_1)\\
&\qquad+b_1(t_1B_2+t_2C_2)+b_2(t_1B_1+t_2C_1)\big)\check s^1\check s^2\\
&+\frac{1}{4}v^6\Big((aC_1-bB_1)\big((-t_1C_2+t_2B_2)\check s^2\check s^3+(t_1B_2+t_2C_2)\check s^2\check s^4\big)\\
&\quad-(-aB_1-bC_1)\big((-t_1C_2+t_2B_2)\check s^1\check s^3+(t_1B_2+t_2C_2)\check s^1\check s^4\big)\\
&\quad+\big((-t_1C_1+t_2B_1)(-t_1C_2+t_2B_2)+(t_1B_1+t_2C_1)(t_1B_2+t_2C_2)\big)\check s^1\check s^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
&E_2\lrcorner X(F_1,E_1,F_2)\lrcorner\Phi=-X(F_1,E_1,F_2)\lrcorner E_2\lrcorner\Phi\\
&=u^2v^4\big((a_2-b_1)(-\bar e^1\check s^4+\bar\nu^4\check s^1-\bar\nu^3\check s^2)+(a_1+b_2)(\bar e^1\check s^3-\bar\nu^4\check s^2-\bar\nu^3\check s^1)\big)\\
&\quad-u^2v^4(a_1a_2+b_1b_1)\omega_1+v^6(a_1a_2+b_1b_2)\check s^1\check s^2\\
&\quad-\frac{1}{2}u^2v^4\big(a_1(-t_1C_2+t_2B_2)+a_2(-t_1C_1+t_2B_1)\\
&\qquad\qquad+b_1(t_1B_2+t_2C_2)+b_2(t_1B_1+t_2C_1)\big)\omega_1\\
&\quad-\frac{1}{2}u^2v^4\left((aC_2-bB_2)(-b_1\omega_2+a_1\omega_3)+(-aB_2-bC_2)(-a_1\omega_2-b_1\omega_3)\right)\\
&\quad+\frac{1}{2}u^2v^4\Big((aC_2-bB_2)(-\bar e^1\check s^2-\bar\nu^4\check s^3+\bar\nu^3\check s^4)+(-aB_2-bC_2)(\bar e^1\check s^1+\bar\nu^4\check s^4+\bar\nu^3\check s^3)\\
&\qquad+\big(t_1(-B_1-C_2)+t_2(B_2-C_1)\big)(-\bar e^1\check s^4+\bar\nu^4\check s^1-\bar\nu^3\check s^2)\\
&\qquad\qquad\qquad+\big(t_1(B_2-C_1)+t_2(B_1+C_2)\big)(\bar e^1\check s^3-\bar\nu^4\check s^2-\bar\nu^3\check s^1)\Big)\\
&\quad-\frac{1}{4}u^2v^4\Big((aC_2-bB_2)\big(-(t_1B_1+t_2C_1)\omega_2+(-t_1C_1+t_2B_1)\omega_3\big)\\
&\qquad+(-aB_2-bC_2)\big(-(-t_1C_1+t_2B_1)\omega_2-(t_1B_1+t_2C_1)\omega_3\big)\\
&\qquad+\big((-t_1C_2+t_2B_2)(-t_1C_1+t_2B_1)+(t_1B_2+t_2C_2)(t_1B_1+t_2C_1)\big)\omega_1\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}v^6\big((aC_2 - bB_2)(a_1\check{s}^2\check{s}^3 + b_1\check{s}^2\check{s}^4) - (-aB_2 - bC_2)(a_1\check{s}^1\check{s}^3 + b_1\check{s}^1\check{s}^4)\big)\\
&+ \frac{1}{2}v^6\big(a_1(-t_1C_2 + t_2B_2) + a_2(-t_1C_1 + t_2B_1)\\
&\qquad\quad + b_1(t_1B_2 + t_2C_2) + b_2(t_1B_1 + t_2C_1)\big)\check{s}^1\check{s}^2\\
&+ \frac{1}{4}v^6\Big((aC_2 - bB_2)\big((-t_1C_1 + t_2B_1)\check{s}^2\check{s}^3 + (t_1B_1 + t_2C_1)\check{s}^2\check{s}^4\big)\\
&\qquad - (-aB_2 - bC_2)\big((-t_1C_1 + t_2B_1)\check{s}^1\check{s}^3 + (t_1B_1 + t_2C_1)\check{s}^1\check{s}^4\big)\\
&\qquad + \big((-t_1C_2 + t_2B_2)(-t_1C_1 + t_2B_1) + (t_1B_2 + t_2C_2)(t_1B_1 + t_2C_1)\big)\check{s}^1\check{s}^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
&- F_1\lrcorner X(E_1, E_2, F_2)\lrcorner\Phi = X(E_1, E_2, F_2)\lrcorner F_1\lrcorner\Phi\\
&\quad = -u^2v^4\big((-a_1 - b_2)(\bar{\nu}^4\check{s}^2 - \bar{e}^1\check{s}^3 - \bar{e}^2\check{s}^4) + (a_2 - b_1)(-\bar{\nu}^3\check{s}^2 + \bar{e}^2\check{s}^3 - \bar{e}^1\check{s}^4)\big)\\
&\quad - \frac{1}{2}u^2v^4\ \big((aC_1 - bB_1)(-a_2\omega_3 + b_2\omega_2) + (aC_2 - bB_2)(a_1\omega_3 - b_1\omega_2)\big)\\
&\quad - \frac{1}{2}u^2v^4\Big(-(aC_1 - bB_1)(\bar{e}^2\check{s}^2 + \bar{\nu}^3\check{s}^3 + \bar{\nu}^4\check{s}^4) - (aC_2 - bB_2)(-\bar{e}^1\check{s}^2 - \bar{\nu}^4\check{s}^3 + \bar{\nu}^3\check{s}^4)\\
&\qquad\quad + \big(t_1(-B_2 + C_1) + t_2(-B_1 - C_2)\big)(\bar{\nu}^4\check{s}^2 - \bar{e}^1\check{s}^3 - \bar{e}^2\check{s}^4)\\
&\qquad\qquad\qquad + \big(t_1(-B_1 - C_2) + t_2(B_2 - C_1)\big)(-\bar{\nu}^3\check{s}^2 + \bar{e}^2\check{s}^3 - \bar{e}^1\check{s}^4)\Big)\\
&\quad - \frac{1}{4}u^2v^4\Big((aC_1 - bB_1)\big(-(-t_1C_2 + t_2B_2)\omega_3 + (t_1B_2 + t_2C_2)\omega_2\big)\\
&\qquad\quad + (aC_2 - bB_2)\big((-t_1C_1 + t_2B_1)\omega_3 - (t_1B_1 + t_2C_1)\omega_2\big)\Big)\\
&\quad + \frac{1}{2}v^6\big((aC_1 - bB_1)(-a_2\check{s}^2\check{s}^3 - b_2\check{s}^2\check{s}^4) + (aC_2 - bB_2)(a_1\check{s}^2\check{s}^3 + b_1\check{s}^2\check{s}^4)\big)\\
&\quad + \frac{1}{4}v^6\Big((aC_1 - bB_1)\big(-(-t_1C_2 + t_2B_2)\check{s}^2\check{s}^3 - (t_1B_2 + t_2C_2)\check{s}^2\check{s}^4\big)\\
&\qquad + (aC_2 - bB_2)\big((-t_1C_1 + t_2B_1)\check{s}^2\check{s}^3 + (t_1B_1 + t_2C_1)\check{s}^2\check{s}^4\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&F_2\lrcorner X(E_2, E_1, F_1)\lrcorner\Phi = -X(E_2, E_1, F_1)\lrcorner F_2\lrcorner\Phi\\
&\quad = -u^2v^4\big((a_2 - b_1)(-\bar{\nu}^4\check{s}^1 + \bar{e}^1\check{s}^4 - \bar{e}^2\check{s}^3) + (a_1 + b_2)(\bar{\nu}^3\check{s}^1 - \bar{e}^2\check{s}^4 - \bar{e}^1\check{s}^3)\big)\\
&\quad - \frac{1}{2}u^2v^4\ \big(-(-aB_1 - bC_1)(a_2\omega_2 + b_2\omega_3) + (-aB_2 - bC_2)(a_1\omega_2 + b_1\omega_3)\big)\\
&\quad - \frac{1}{2}u^2v^4\Big((-aB_1 - bC_1)(-\bar{e}^2\check{s}^1 - \bar{\nu}^3\check{s}^4 + \bar{\nu}^4\check{s}^3) + (-aB_2 - bC_2)(\bar{e}^1\check{s}^1 + \bar{\nu}^4\check{s}^4 + \bar{\nu}^3\check{s}^3)\\
&\qquad\quad + \big(t_1(-B_1 - C_2) + t_2(B_2 - C_1)\big)(-\bar{\nu}^4\check{s}^1 + \bar{e}^1\check{s}^4 - \bar{e}^2\check{s}^3)\\
&\qquad\qquad\qquad + \big(t_1(B_2 - C_1) + t_2(B_2 + C_2)\big)(\bar{\nu}^3\check{s}^1 - \bar{e}^2\check{s}^4 - \bar{e}^1\check{s}^3)\Big)\\
&\quad - \frac{1}{4}u^2v^4\Big(-(-aB_1 - bC_1)\big((-t_1C_2 + t_2B_2)\omega_2 + (t_1B_2 + t_2C_2)\omega_3\big)\\
&\qquad\quad + (-t_1C_1 + t_2B_1)(-aB_2 - bC_2)\omega_2 + (t_1B_1 + t_2C_1)(-aB_2 - bC_2)\omega_3\Big)\\
&\quad - \frac{1}{2}v^6\big(-(-aB_1 - bC_1)(-a_2\check{s}^1\check{s}^3 - b_2\check{s}^1\check{s}^4) + (-aB_2 - bC_2)(-a_1\check{s}^1\check{s}^3 - b_1\check{s}^1\check{s}^4)\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}v^6\Big(-(-aB_1-bC_1)\big(-(-t_1C_2+t_2B_2)\check{s}^1\check{s}^3-(t_1B_2+t_2C_2)\check{s}^1\check{s}^4\big)\\
&\qquad\qquad -(-aB_2-bC_2)\big((-t_1C_1+t_2B_1)\check{s}^1\check{s}^3+(t_1B_1+t_2C_1)\check{s}^1\check{s}^4\big)\Big).
\end{aligned}
$$

These formulas add up to

$$
\begin{aligned}
&E_1 \lrcorner X(E_2,F_1,F_2)\lrcorner\Phi + E_2\lrcorner X(F_1,E_1,F_2)\lrcorner\Phi + F_1\lrcorner X(E_1,E_2,F_2)\lrcorner\Phi + F_2\lrcorner X(E_2,E_1,F_1)\lrcorner\Phi\\
&\quad = 3u^2v^4\big((a_1+b_2)(\bar{e}^1\check{s}^3+\bar{e}^2\check{s}^4-\bar{\nu}^3\check{s}^1-\bar{\nu}^4\check{s}^2)+(a_2-b_1)(-\bar{e}^1\check{s}^4+\bar{e}^2\check{s}^3-\bar{\nu}^3\check{s}^2+\bar{\nu}^4\check{s}^1)\big)\\
&\qquad +\frac{3}{2}u^2v^4\Big(\big(t_1(B_2-C_1)+t_2(B_1+C_2)\big)(\bar{e}^1\check{s}^3+\bar{e}^2\check{s}^4-\bar{\nu}^3\check{s}^1-\bar{\nu}^4\check{s}^2)\\
&\qquad\qquad +\big(t_1(B_1+C_2)+t_2(-B_2+C_1)\big)(\bar{e}^1\check{s}^4-\bar{e}^2\check{s}^3+\bar{\nu}^3\check{s}^2-\bar{\nu}^4\check{s}^1)\Big).
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&\eta(E_1,E_2,F_1,F_2)\\
&\quad = 2u^2v^4\big((a_1+b_2)(\bar{e}^1\check{s}^3+\bar{e}^2\check{s}^4-\bar{\nu}^3\check{s}^1-\bar{\nu}^4\check{s}^2)+(a_2-b_1)(-\bar{e}^1\check{s}^4+\bar{e}^2\check{s}^3-\bar{\nu}^3\check{s}^2+\bar{\nu}^4\check{s}^1)\big)\\
&\qquad +u^2v^4\Big(\big(t_1(B_2-C_1)+t_2(B_1+C_2)\big)(\bar{e}^1\check{s}^3+\bar{e}^2\check{s}^4-\bar{\nu}^3\check{s}^1-\bar{\nu}^4\check{s}^2)\\
&\qquad\qquad +\big(t_1(B_1+C_2)+t_2(-B_2+C_1)\big)(\bar{e}^1\check{s}^4-\bar{e}^2\check{s}^3+\bar{\nu}^3\check{s}^2-\bar{\nu}^4\check{s}^1)\Big).
\end{aligned}
$$

B. Spin geometry

This section briefly reviews the basics of spin geometry required to fully understand the constructions in the negative spinor bundles $\$_-(X)$ of $X^4 = \mathbb{R}^4, S^4$. We focus on presenting the main ideas and refer to [Har90, Ch. 9–11], [Wen22, Sec. 50] and [LM89, Ch. 1–2] for more details.

Let V be an n-dimensional vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, and denote its tensor algebra by $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$. The **Clifford algebra** $\mathrm{Cl}(V)$ of V is defined as

$$\mathrm{Cl}(V) = T(V)/I,$$

where $I \subset T(V)$ is the two-sided ideal generated by $\{v \otimes v + \langle v, v \rangle \mid v \in V\}$. We write $x \cdot y = [v \otimes w] \in \mathrm{Cl}(V)$ for the product of $x = [v]$ and $y = [w] \in \mathrm{Cl}(V)$.

As a vector space, $\mathrm{Cl}(V)$ is naturally isomorphic to the exterior algebra $\Lambda^* V = \bigoplus_{k=0}^{\infty} \Lambda^k V$. Specifically, after embedding $\Lambda^* V$ into $T(V)$ via

$$v_1 \dot{\wedge} \dots \dot{\wedge} v_k = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}$$

for $v_1, \dots, v_k \in V$, the quotient map $\Lambda^* V \subset T(V) \to \mathrm{Cl}(V) = T(V)/I$ serves as a vector space isomorphism [Har90, Prop. 9.11]. In particular, the natural maps $\mathbb{R} = V^{\otimes 0} \to \mathrm{Cl}(V)$ and $V = V^{\otimes 1} \to \mathrm{Cl}(V)$ are injective, allowing us to regard $\mathbb{R}$ and V as subspaces of $\mathrm{Cl}(V)$.

Under this map, the Clifford product $\cdot$ on $\mathrm{Cl}(V)$ and the wedge product $\dot{\wedge}$ on $\Lambda^* V$ are related by

$$v \cdot w = v \dot{\wedge} w - v \lrcorner w$$

for $v \in V$ and $w \in \Lambda^* V \cong \mathrm{Cl}(V)$ [Har90, Prop. 9.11], which simplifies to

$$v_1 \cdot \dots \cdot v_k = v_1 \dot{\wedge} \dots \dot{\wedge} v_k = \frac{1}{k!} v_1 \wedge \dots \wedge v_k \tag{B.1}$$

for orthogonal vectors $v_1, \dots, v_k \in V$. Here, $\wedge$ denotes the wedge product according to the convention used in this thesis, given by

$$v_1 \wedge \dots \wedge v_k = \sum_{\sigma \in S_k} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}$$

for $v_1, \dots, v_k \in V$ [Spi99, Ch. 7], [Wen22, Sec. 9].

Through the canonical vector space isomorphism $\mathrm{Cl}(V) \cong \Lambda^* V$, the Clifford algebra $\mathrm{Cl}(V)$ also inherits the inner product from $\Lambda^* V$, which is defined by

$$\langle v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_l \rangle = \begin{cases} \det(\langle v_i, w_j \rangle), & k = l, \\ 0, & k \neq l. \end{cases} \tag{B.2}$$

Let us now fix an orthonormal basis $e_1, \dots, e_n$ for V. Equivalently, we could have defined $\mathrm{Cl}(V)$ to be the algebra generated by $e_1, \dots, e_n$ subject to the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}. \tag{B.3}$$

As a vector space, $\mathrm{Cl}(V)$ is spanned by the elements of the form $e_1^{\delta_1} \cdot \ldots \cdot e_n^{\delta_n}$ for $\delta_i \in \{0,1\}$, which are orthogonal by (B.1) and (B.2). This allows us to split $\mathrm{Cl}(V) = \mathrm{Cl}^{\mathrm{even}}(V) \oplus \mathrm{Cl}^{\mathrm{odd}}(V)$ into an **even** and an **odd part**, given by

$$\mathrm{Cl}^{\mathrm{even}}(V) = \mathrm{span}\Big\{ e_1^{\delta_1} \cdot \ldots \cdot e_n^{\delta_n} \;\Big|\; \delta_i \in \{0,1\},\ \sum_{i=1}^n \delta_i \in 2\mathbb{Z} \Big\},$$
$$\mathrm{Cl}^{\mathrm{odd}}(V) = \mathrm{span}\Big\{ e_1^{\delta_1} \cdot \ldots \cdot e_n^{\delta_n} \;\Big|\; \delta_i \in \{0,1\},\ \sum_{i=1}^n \delta_i \in 2\mathbb{Z}+1 \Big\}.$$

Within the even part, we find the **spin group** of V, which is defined as

$$\mathrm{Spin}(V) = \{ v_1 \cdot \ldots \cdot v_{2N} \mid N \geq 0,\ v_i \in V,\ \|v_i\| = 1 \} \subset \mathrm{Cl}^{\mathrm{even}}(V).$$

In fact, $\mathrm{Spin}(V)$ is a Lie group and there exists a natural covering map $\Phi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$, $h \mapsto \mathrm{Ad}_h|_V$ of degree 2, given by the restriction of the **adjoint representation**

$$\mathrm{Ad} : \mathrm{Spin}(V) \to \mathrm{GL}(\mathrm{Cl}(V)), \quad \mathrm{Ad}_h(y) = h \cdot y \cdot h^{-1}$$

to $V \subset \mathrm{Cl}(V)$ [Wen22, Thm. 50.15].

We refer to $\lambda = e_1 \cdot \ldots \cdot e_n \in \mathrm{Cl}(V)$ as the **volume element**. Using (B.3), we calculate $\lambda^2 = (-1)^{n(n+1)/2}$, which equals $+1$ whenever $n \in \{0,3\} \mod 4$. For such n, λ induces a natural splitting of $\mathrm{Cl}(V) = \mathrm{Cl}_+(V) \oplus \mathrm{Cl}_-(V)$ into a **self-dual** and an **anti-self-dual part**, given by its eigenspaces

$$\mathrm{Cl}_\pm(V) = \{ y \in \mathrm{Cl}(V) \mid \lambda \cdot y = \pm y \}.$$

Let us now restrict our attention to $V = \mathbb{R}^n$ with the standard inner product, and denote the associated Clifford algebra by $\mathrm{Cl}(n)$. **Pinor** and **spinor representations** are irreducible representations of $\mathrm{Cl}(n)$ and $\mathrm{Cl}^{\mathrm{even}}(n)$, respectively. Specifically for $n = 4 \mod 8$, these terms refer to irreducible $\mathbb{H}$-representations

$$\gamma : \mathrm{Cl}(n) \cong \mathrm{End}_{\mathbb{H}}(\mathcal{P}) \qquad \text{and} \qquad \rho_\pm : \mathrm{Cl}_\pm^{\mathrm{even}} \cong \mathrm{End}_{\mathbb{H}}(\$_\pm)$$

for $\mathbb{H}$-vector spaces $\mathcal{P}$ (**space of pinors**) and $\$ = \$_+ \oplus \$_-$ (**space of** (**positive** and **negative**) **spinors**) [Har90, Def. 11.10, 11.14]. As $\gamma(\lambda)$ satisfies $\gamma(\lambda)^2 = \gamma(\lambda^2) = 1$, we can split $\mathcal{P} = \mathcal{P}_+ \oplus \mathcal{P}_-$ into its eigenspaces

$$\mathcal{P}_\pm = \{ a \in \mathcal{P} \mid \gamma(\lambda) a = \pm a \}.$$

Then the pinor representation takes the form

$$\gamma : \mathrm{Cl}(n) \cong \mathrm{End}_{\mathbb{H}}(\mathcal{P}) = \mathrm{End}_{\mathbb{H}}(\mathcal{P}_+ \oplus \mathcal{P}_-) \cong \begin{pmatrix} \mathrm{End}_{\mathbb{H}}(\mathcal{P}_+) & \mathrm{Hom}_{\mathbb{H}}(\mathcal{P}_+, \mathcal{P}_-) \\ \mathrm{Hom}_{\mathbb{H}}(\mathcal{P}_-, \mathcal{P}_+) & \mathrm{End}_{\mathbb{H}}(\mathcal{P}_-) \end{pmatrix}.$$

Since $\gamma(y)$ preserves $\mathcal{P}_\pm$ for $y \in \mathrm{Cl}^{\mathrm{even}}(n)$ and interchanges them for $y \in \mathrm{Cl}^{\mathrm{odd}}(n)$, the restriction of γ to $\mathrm{Cl}^{\mathrm{even}}(n)$ yields the irreducible representation

$$\gamma|_{\mathrm{Cl}^{\mathrm{even}}(n)} : \mathrm{Cl}^{\mathrm{even}}(n) \cong \begin{pmatrix} \mathrm{End}_{\mathbb{H}}(\mathcal{P}_+) & 0 \\ 0 & \mathrm{End}_{\mathbb{H}}(\mathcal{P}_-) \end{pmatrix} \cong \mathrm{End}_{\mathbb{H}}(\mathcal{P}_+) \oplus \mathrm{End}_{\mathbb{H}}(\mathcal{P}_-).$$

Due to the relation $\gamma(\lambda)\gamma(y) = \gamma(\lambda \cdot y) = \pm\gamma(y)$ for $y \in \mathrm{Cl}^{\mathrm{even}}_{\pm}(n)$, this leads to the spinor representation

$$\rho_\pm = \gamma|_{\mathrm{Cl}^{\mathrm{even}}_{\pm}(n)} : \mathrm{Cl}^{\mathrm{even}}_{\pm}(n) \cong \mathrm{End}_{\mathbb{H}}(\mathcal{P}_\pm).$$

Thus, $\$_\pm = \mathcal{P}_\pm$ and $\$ = \mathcal{P}$.

From the equality $\dim \mathrm{Cl}^{\mathrm{even}}(n) = \dim \mathrm{Cl}^{\mathrm{odd}}(n)$, it follows that

$$\begin{aligned} 0 &= \dim_{\mathbb{H}}\big(\mathrm{End}_{\mathbb{H}}(\$_+)\big) + \dim_{\mathbb{H}}\big(\mathrm{End}_{\mathbb{H}}(\$_-)\big) \\ &\quad - \dim_{\mathbb{H}}\big(\mathrm{Hom}_{\mathbb{H}}(\$_+, \$_-)\big) - \dim_{\mathbb{H}}(\mathrm{Hom}_{\mathbb{H}}(\$_-, \$_+)) \\ &= (\dim_{\mathbb{H}} \$_+)^2 + (\dim_{\mathbb{H}} \$_-)^2 - 2(\dim_{\mathbb{H}} \$_+)(\dim_{\mathbb{H}} \$_-) \\ &= (\dim_{\mathbb{H}} \$_+ - \dim_{\mathbb{H}} \$_-)^2, \end{aligned}$$

which shows that $\dim_{\mathbb{H}} \$_+ = \dim_{\mathbb{H}} \$_-$. Furthermore, we have

$$\mathrm{Cl}^{\mathrm{even}}(n) \cong M_N(\mathbb{H}) \oplus M_N(\mathbb{H})$$

for $N = 2^{n/2-2}$ [Har90, Cor. 11.8].

Combining all of this in the case of $n = 4$ and restricting ρ to a representation of $\mathrm{Spin}(n)$, we obtain the pinor and spinor representations

$$\begin{aligned} \gamma &: \mathrm{Cl}(4) \cong \mathrm{End}_{\mathbb{H}}(\$_+ \oplus \$_-), \\ \rho = \rho_+ \oplus \rho_- &: \mathrm{Spin}(4) \subset \mathrm{Cl}^{\mathrm{even}}(4) \cong \mathrm{End}_{\mathbb{H}}(\$_+) \oplus \mathrm{End}_{\mathbb{H}}(\$_-) \end{aligned}$$

with $\$_\pm \cong \mathbb{H}$.

Now let (X^4, g) be an oriented Riemannian manifold with **spin structure** $(P_{\mathrm{Spin}}, \Psi)$, consisting of a principal $\mathrm{Spin}(4)$-bundle $P_{\mathrm{Spin}} \to X$ and a smooth fiber-preserving 2-fold covering map $\Psi : P_{\mathrm{Spin}} \to F^{\mathrm{SO}(4)}(TX)$ such that $\Psi(ph) = \Psi(p)\Phi(h)$ for all $p \in P_{\mathrm{Spin}}$ and $h \in \mathrm{Spin}(4)$. In other words, P_{Spin} is an equivariant lift of the oriented orthonormal frame bundle $F^{\mathrm{SO}(4)}(TX) \to X$ with respect to the double cover $\Phi : \mathrm{Spin}(4) \to \mathrm{SO}(4)$. The **spinor bundle** of X is defined as

$$\$(X) = P_{\mathrm{Spin}} \times_\rho \$ = (P_{\mathrm{Spin}} \times \$)/\mathrm{Spin}(4) \to X.$$

This is a quaternionic vector bundle of rank 2 with a natural $\mathrm{Spin}(4)$-structure and the same transition functions as P_{Spin}, but with standard fiber $\$ = \$_+ \oplus \$_- \cong \mathbb{H} \oplus \mathbb{H}$. The bundle $\$(X) = \$_+(X) \oplus \$_-(X)$ splits into the **positive** and **negative spinor bundles**, which are given by the associated bundles

$$\$_\pm(X^4) = P_{\mathrm{Spin}} \times_{\rho_\pm} \$_\pm = (P_{\mathrm{Spin}} \times \$_\pm)/\mathrm{Spin}(4) \to X$$

with standard fiber $\$_\pm \cong \mathbb{H}$.

The (positive/negative) spinor bundle possesses a canonical connection, which we call the **spin connection**. Indeed, on the tangent bundle $TX \to X$, the natural choice is the Levi-Civita connection. This induces a principal connection on the principal $\mathrm{SO}(4)$-bundle $F^{\mathrm{SO}(4)}(TX)$. Demanding that the parallel transport map on the principal $\mathrm{Spin}(4)$-bundle P_{Spin} commutes with the double cover $\Psi : P_{\mathrm{Spin}} \to F^{\mathrm{SO}(4)}(TX)$ uniquely determines a principal connection on P_{Spin}. Finally, the latter induces Spin(4)-compatible connections on the associated bundles $\$(X) \to X$ and $\$_{\pm}(X) \to X$.

Next, we introduce the **Clifford bundle**

$$\text{Cl}(TX) = \bigcup_{x \in X} \text{Cl}(T_x X),$$

whose fiber over $x \in X$ is the Clifford algebra of $T_x X$ with its inner product g_x. Equivalently, we can define it as the associated bundle

$$\text{Cl}(TX) = P_{\text{Spin}} \times_{\text{Ad}} \text{Cl}(4) = (P_{\text{Spin}} \times \text{Cl}(4))/\text{Spin}(4) \to X$$

with standard fiber Cl(4) (see [Wen22, Sec. 50.5] for more details).

Since the linear map $\text{Cl}(4) \otimes \$ \to \$: y \otimes s \mapsto \gamma(y)s$ is Spin(4)-equivariant:

$$h(y \otimes s) = (h \cdot y \cdot h^{-1}) \otimes (\gamma(h)s) \mapsto \gamma(h \cdot y \cdot h^{-1})(\gamma(h)s) = \gamma(h)(\gamma(y)s) = h(\gamma(y)s)$$

for $h \in \text{Spin}(4)$, $y \in \text{Cl}(4)$ and $s \in \$$, it induces a smooth linear bundle map

$$\text{Cl}(TX) \otimes \$(X) \to \$(X) : y \otimes s \mapsto y \cdot s,$$

called **Clifford multiplication** on $\$(X)$. This can be interpreted as a bilinear bundle map $\text{Cl}(TX) \oplus \$(X) \to \$(X)$ and turns each fiber $\$_x(X)$ into a $\text{Cl}(T_x X)$-module.

One can check that the spin connection on $\$(X) \to X$ is compatible with the Clifford multiplication on $\$(X)$ and the Levi-Civita connection on $TX \to X$ in the sense that

$$\nabla_u(v \cdot s) = (\nabla_u v) \cdot s + v \cdot \nabla_u s$$

for all $u, v \in \Gamma(TX)$ and $s \in \Gamma(\$(X))$ [Wen22, Ex. 50.19].

C. Octonion multiplication table

The octonion multiplication rules are captured in the following table. A cell represents the product of the element in the corresponding row (on the left) and the element in the corresponding column (on the right). For example, $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$.

·	1	i	j	k	e	ie	je	ke
1	1	i	j	k	e	ie	je	ke
i	i	-1	k	-j	ie	-e	-ke	je
j	j	-k	-1	i	je	ke	-e	-ie
k	k	j	-i	-1	ke	-je	ie	-e
e	e	-ie	-je	-ke	-1	i	j	k
ie	ie	e	-ke	je	-i	-1	-k	j
je	je	ke	e	-ie	-j	k	-1	-i
ke	ke	-je	ie	e	-k	-j	i	-1

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