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Theorem von drei geodätischen Kurven unter Verwendung des Kurvenverkürzungsflusses

Three Geodesics Theorem Using Curve Shortening Flow

Bachelorarbeit

zur Erlangung des akademischen Grades

Bachelor of Science (B.sc.)

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Abstract

The three geodesics theorem states that on the 2-sphere equipped with any Riemannian metric, there exist at least three simple closed geodesics. The initial attempts on proving this theorem relied on the theory of critical points. Although the general topological theory in the proofs were very convincing, there were still flaws in the constructions of the deformations.

The curve shortening flow was first studied in the 1980s by several mathematicians like Gage, Hamilton and Grayson. In the late 1980s, Grayson gave an universally accepted proof of the three geodesics theorem by using this flow. This thesis aims to prove the three geodesics theorem following Grayson's approach, with an emphasis on introducing the curve shortening flow.

Chapter 1

Introduction

In 1929, Lusternik and Schnirelmann [1] first claimed the three geodesics theorem as below:

Main Theorem 1.1 (Three geodesics theorem, 1929)**.** *On every Riemannian manifold (S^2, g) there exist at least three simple closed geodesics, where g is a Riemannian metric.*

They also provided a proof using a algebraic topological argument, but later the deformation in the proof was discovered to be wrong. Since then many mathematicians have given the attempts on filling the gap in the original proof, like Klingenberg [5] and Ballmann [3]. In the proofs they considered the geodesics as critical points of the energy functional (see [2], [5]. and [6]). They looked for the non-zero homology classes of the space of simple closed curves on (S^2, g) and then tried to shorten the cycles representing each homology class to obtain the critical points. Therefore the challenge is how to shorten a curve, while not losing some properties of it during the deformation.

In the 1980s, mathematicians started to study the curve shortening flow. The flow obtained its name because it shortens the curve as fast as possible. Gage and Hamilton showed in [7] that any convex curve evolved by the curve shortening flow will shrink to a point with round limiting shape. One year later, in [8] Grayson removed the convexity in the assumption and proved that only the embeddedness is required. In 1989, Grayson [9] added that if a curve evolves under the flow for infinite time, then it will converge to a geodesic. He combined this result with the works of Ballmann, Thorbergsson and Ziller [6] and proved the three geodesics theorem.

This thesis will proceed as follows:

- In Chapter 2 we begin with some basic geometry of planar curves, then we introduce the curve shortening flow and study its basic properties. From this we prove more properties and theorems, then we will be in the position to prove the Grayson's theorem and obtain the convergence to geodesics. We will follow the book [15] in the first four sections, the last section is based on [9].

- In Chapter 3 we move to the algebraic topological part. We will first define the "loop space" and use curve shortening flow to find suitable deformation retracts for it. Given the retracts we can determine the homology of the loop space relative to the space of point curves. Last, we complete the proof of the main theorem with critical point theory. We will move along following [9] and [4] in this chapter, also with results in [6] and inspirations from [14].

Chapter 2

Curve Shortening Flow

The **curve shortening flow** is a special (1-dimensional) case of the **mean curvature flow**. It moves every single point on a planar curve in the direction of the inward pointing normal vector so that the curve length will be decreased in the most efficient way.
In this thesis we denote a compact 1-manifold by M^1. We begin with basic geometry in $\mathbb{R}^2$, then we introduce the curve shortening flow and some important properties of it. Later we prove the Grayson's theorem to deduce the convergence of an embedded curve to a geodesic, if its evolution under the curve shortening flow lasts forever.

2.1 Basic Geometry of Planar Curves

From now on let X be a curve, s be an arc length parameter such that $\frac{d}{ds} := \frac{1}{|X'|}\frac{d}{du}$. Furthermore we can define:

Definition 2.1. *The **unit tangent vector** $T := \frac{dX}{ds}$ and the **unit normal vector** (pointing outwards) N is obtained by rotating T counter-clockwise through angle $\pi/2$.*

Definition 2.2. *The **curvature** κ of a curve X is defined as*

$$\kappa = -\left\langle \frac{dT}{ds},\ N \right\rangle = \left\langle \frac{dN}{ds},\ T \right\rangle. \tag{2.1}$$

Remark 2.3. *This definition is actually derived from the Frenet-Serret formulae:*

$$\frac{dT}{ds} = -\kappa N,$$
$$\frac{dN}{ds} = \kappa T.$$

Definition 2.4. *The **normal angle** θ of the curve is the function $\theta : M^1 \to \mathbb{R}$ such that*

$$N(u) = (\cos\theta(u), \sin\theta(u))$$

for every $u \in M^1$.

Due to the relation between T and N we have

$$T(u) = (-\sin\theta(u), \cos\theta(u)).$$

Then we compute the curvature:

$$\begin{aligned}
\kappa &= -\left\langle \frac{d}{ds}(-\sin\theta(p), \cos\theta(p)),\ (\cos\theta(p), \sin\theta(p)) \right\rangle \\
&= -\left\langle \frac{d\theta}{ds}(-\cos\theta(p), -\sin\theta(p)),\ (\cos\theta(p), \sin\theta(p)) \right\rangle \\
&= \frac{d\theta}{ds}\left(\cos^2\theta(p) + \sin^2\theta(p)\right) = \frac{d\theta}{ds}
\end{aligned} \tag{2.2}$$

Definition 2.5. *The smooth 1-form on M^1 which is dual to $\frac{d}{ds}$ is called the **arc length element** and is denoted by ds. Furthermore we define the **length** of a curve as*

$$L := \int_{M^1} ds.$$

2.2 Properties of the Curve Shortening Flow

For simplification, we introduce the new notations: $f_s := \frac{\partial f}{\partial_s}$ and $f_t := \frac{\partial f}{\partial_t}$ for a given function f.

Definition 2.6. *Let $X(\cdot\,, t) : M^1 \times [0, T) \to \mathbb{R}^2$ be a smooth 1-parameter family of immersions. X is said to be **evolved by the curve shortening flow** or called a **solution to the curve shortening flow** if it solves the parabolic partial differential equation*

$$X_t = -\kappa N. \tag{2.3}$$

Remark 2.7. *We have $X_t = -\kappa N = T_s = (X_s)_s = X_{ss}$, so this flow can also be considered as a non-linear heat-type equation.*

Lemma 2.8. *Let X be a solution to the curve shortening flow, then for every $f : M^1 \times [0, T) \to \mathbb{R}$, $f \in C^2$ we have*

$$f_{st} = f_{ts} + \kappa^2 f_s. \tag{2.4}$$

Proof. First compute that

$$\begin{aligned}\langle X_{tu},\ X_u\rangle &= \langle X_{ut},\ |X_u|T\rangle = \langle (-\kappa N)_u\,,\ |X_u|T\rangle = \langle -\kappa_u N - \kappa N_u,\ |X_u|T\rangle \\ &= \langle -\kappa_u N,\ |X_u|T\rangle + \langle -\kappa N_u,\ |X_u|T\rangle = \left\langle -\kappa^2|X_u|T,\ |X_u|T\right\rangle \\ &= -|X_u|^2\kappa^2\end{aligned}$$

We used in the first step $X_{tu} = X_{ut}$ since t, u are two independent parameters, and $\langle N,\ T\rangle = 0$, $N_u = |X_u|N_s = \kappa|X_u|T$ in the last two steps.
Now we have

$$\begin{aligned}f_{st} &= \left(f_u|X_u|^{-1}\right)_t \\ &= f_{tu}|X_u|^{-1} - |X_u|^{-1}|X_u|^{-2}\langle X_{tu},\ X_u\rangle f_u \\ &= f_{ts} + |X_u|^{-3}|X_u|^2\kappa^2 f_u \\ &= f_{ts} + \kappa^2 f_s.\end{aligned}$$

□

Lemma 2.9. *Let X be a solution to the curve shortening flow, then it has the following properties:*

(i) $(ds)_t = -\kappa^2 ds$.

(ii) $T_t = -\kappa_s N$.

(iii) $N_t = \kappa_s T$.

(iv) $\theta_t = \kappa_s$.

Proof. (i) Note that

$$(ds)_t^2 = 2ds(ds)_t, \quad (\frac{ds}{du})_t^2 = \frac{1}{(du)^2}(ds)_t^2.$$

Moreover,

$$\begin{aligned}(\frac{ds}{du})_t^2 &= |X_u|_t^2 = 2\langle \partial_t X_u,\ X_u\rangle = 2\langle X_{ut},\ X_u\rangle = 2\langle X_u,\ X_{tu}\rangle \\ &= 2\langle |X_u|T,\ \partial_u(-\kappa N)\rangle = -2\langle |X_u|T,\ \kappa_u N + \kappa N_u\rangle \\ &= -2\left\langle |X_u|T,\ \kappa_u N + \kappa^2|X_u|T\right\rangle = -2\kappa^2|X_u|^2 \\ &= -2\kappa^2(\frac{ds}{du})^2.\end{aligned}$$

Then

$$(ds)_t = \frac{-\kappa^2(ds)_t^2}{2ds} = -\kappa^2 ds.$$

(ii) Lemma 2.6 implies

$$T_t = X_{st} = X_{ts} + \kappa^2 X_s = (-\kappa N)_s + \kappa^2 T = -\kappa_s N - \kappa\kappa TT + \kappa^2 T = -\kappa_s N$$

(iii) Consider

$$0 = \langle N,\ T\rangle_t = \langle N_t,\ T\rangle + \langle N,\ T_t\rangle = \langle N_t,\ T\rangle + \langle N,\ -\kappa_s N\rangle\,,$$

then N_t must be equal to $\kappa_s T$. In the last equality we used (ii).

(iv) Due to (ii) and the definition of T we have

$$\theta_t\left(-\cos\theta(u), -\sin\theta(u)\right) = T_t = -\kappa_s N = -\kappa_s\left(\cos\theta(u), \sin\theta(u)\right),$$

comparing the components the equality is proved.

□

Lemma 2.10. *Under the curve shortening flow, κ evolves by*

$$\kappa_t = \kappa_{ss} + \kappa^3. \tag{2.5}$$

Proof. It follows from Lemma 2.8, Lemma 2.9(iv) and $\theta_s = \kappa$. □

Combining this lemma with the maximum principle we can get one important property, which states that the convexity of a curve will be preserved under the curve shortening flow.

Corollary 2.11. *Let X be evolved by the curve shortening flow, if X is initially convex, then it will remain convex during the evolution.*

Proof. Let $\kappa_m(t)$ denote the minimal curvature at $u \in X$ at time t,then $\kappa_m(0) > 0$. Consider the PDE with initial data:

$$\frac{d\phi}{dt} = \phi^3,$$
$$\phi(0) = \phi_0.$$

Applying the weak maximum principle for non-linear PDEs, we know that $\kappa_m \geq \varphi(t)$, where $\varphi(t) = \frac{\kappa_m(0)}{\sqrt{1-2t(\kappa_m(0))^2}} \geq 0$ the solution to the above PDE. We conclude that $\kappa_m \geq 0$ and this means the curve remains convex at any time t. □

Now we can find the evolutions of the length of a curve and also the area enclosed by a simple closed curve.

Lemma 2.12. $L_t = -\int_{M^1} \kappa^2 ds.$

Proof. $L = \int_{M^1} ds$ and Lemma 2.9(i) implies $L_t = \int_{M^1} (ds)_t = -\int_{M^1} \kappa^2 ds$. □

Lemma 2.13. $A_t = -2\pi.$

Proof.

$$A = \frac{1}{2}\int_{M^1} (xy_u - yx_u)du = -\frac{1}{2}\int_{M^1} \langle X,\ |X_u|N\rangle\, du = -\frac{1}{2}\int_{M^1} \langle X,\ N\rangle\, ds.$$

We rearrange this relation and differentiate both sides with respect to time,

$$\begin{aligned} 2A_t &= \int_{M^1} \left(\left(\langle X_t,\ N\rangle + \langle X,\ N_t\rangle\right) ds + \langle X,\ N\rangle\, ds_t\right) \\ &= \int_{M^1} \big(\left(\langle -\kappa N,\ N\rangle + \langle X,\ \kappa_s T\rangle\right) - \kappa_s^2 \langle X,\ N\rangle\big)\, ds \\ &= \int_{M^1} \big(\left(-\kappa + \kappa_s \langle X,\ T\rangle\right) - \kappa_s^2 \langle X,\ N\rangle\big)\, ds \end{aligned}$$

Integrating by parts yields

$$\int_{M^1} \kappa_s \langle X,\ T\rangle\, ds = \int_{M^1} \big(-\kappa + \kappa^2 \langle X,\ T\rangle\big)\, ds.$$

Thus by $\kappa T = N_s$,

$$A_t = \frac{1}{2}\int_{M^1} -\kappa ds = -2\pi.$$

□

Remark 2.14. *Lemma 2.12 and Lemma 2.13 tell us that the curve shortening flow shrinks the length as well as the area of a curve. Additionally, Lemma 2.13 implies a bound on the maximal existence time (denoted by T_{max}) of a solution: $T_{max} \leq \frac{A(0)}{2\pi}$, since the area $A(t) = A(0) - 2\pi t$ has to be non-negative.*

2.3 Existence of the Curve Shortening Flow

For further study we have to show that a solution to the curve shortening flow does exist. In this section we are going to analyse its existence both locally and globally. The short-time (local) existence is deduced from the short-time existence of the mean curvature flow. For the short-time existence theorem and its proof for the mean curvature flow, see [15, Chapter 6].

Theorem 2.15 (Short-time Existence)**.** *Let X_0 be a smooth immersion. Then there exists a smooth solution to the curve shortening flow $X(t)$ with $X(0) = X_0$ on time interval $[0, T)$, $T > 0$.*

Gage and Hamilton first showed in [7] the solution to the non-linear parabolic equation exists globally by using a priori estimates. But here we will follow the proof of the long-term existence in [15], for that we need to find some curvature estimates.

Lemma 2.16. *If there exists a solution to the curve shortening flow on a closed curve and κ is bounded by K on the time interval $[0, t_0)$ with $0 < t_0 < K^{-2}$. Then κ_s is also bounded on $(0, t_0)$ by $CKt^{-\frac{1}{2}}$, where $C > 0$ is a constant.*

Proof. By Lemma 2.8 and Lemma 2.10 the time derivative of κ_s is

$$\kappa_{st} = \kappa_{ts} + \kappa^2\kappa_s = \kappa_{sss} + 4\kappa^2\kappa_s. \tag{2.6}$$

Differentiating $(\kappa_s)^2$ we have

$$((\kappa_s)^2)_t = 2\kappa_s\kappa_{st} = 2\kappa_s(\kappa_{sss} + 4\kappa^2\kappa_s) = 8\kappa^2(\kappa_s)^2 + ((\kappa_s)^2)_{ss} - 2(\kappa_{ss}^2). \tag{2.7}$$

And again by Lemma 2.10

$$(\kappa^2)_t = 2\kappa\kappa_t = 2\kappa(\kappa_{ss} + \kappa^3) = 2\kappa^4 + (\kappa^2)_{ss} - 2(\kappa_s)^2. \tag{2.8}$$

Now let α be a positive constant, combining (2.6), (2.7) and (2.8) yields

$$\begin{aligned}(t(\kappa_s)^2 + \alpha\kappa^2)_t - (t(\kappa_s)^2 + \alpha\kappa^2)_{ss} &= (\kappa_s)^2 + 2t\kappa_s\kappa_{st} + 2\alpha\kappa\kappa_t \\ &\quad - 2t\kappa_s\kappa_{sss} - 2\alpha\kappa\kappa_{ss} - 2\alpha(\kappa_s)^2 \\ &= -2t(\kappa_{ss})^2 + 2\alpha\kappa(\kappa_t - \kappa_{ss}) + (1 - 2\alpha)(\kappa_s)^2 \\ &\quad + 2t\kappa_s(\kappa_{st} - \kappa_{sss}) \\ &= -2t(\kappa_{ss})^2 + 2\alpha\kappa^4 + (1 - 2\alpha)(\kappa_s)^2 \\ &\quad + 2t\kappa_s(4\kappa^2\kappa_s) \\ &= -2t(\kappa_{ss})^2 + 2\alpha\kappa^4 + (\kappa_s)^2(-2\alpha + 1 + 8t\kappa^2).\end{aligned} \tag{2.9}$$

We choose some α satisfying $-2\alpha + 1 + 8t\kappa^2 = 0$, then

$$(t(\kappa_s)^2 + \alpha\kappa^2)_t - (t(\kappa_s)^2 + \alpha\kappa^2)_{ss} = -2t(\kappa_{ss})^2 + 2\alpha\kappa^4 \leq \beta K^4. \tag{2.10}$$

for some constant $\beta > 0$. At $t = 0$, we have $t(\kappa_s)^2 + \alpha\kappa^2 \leq \alpha K^2$. Applying the maximum principle we have

$$t(\kappa_s)^2 + \alpha\kappa^2 \leq \beta K^2 + \alpha K^2.$$

for $t \in [0, t_0)$. Then there is a positive constant C such that

$$|\kappa_s| \leq CKt^{\frac{-1}{2}}.$$

$\square$

The higher derivatives estimates are also important, the following theorem states that there exist some similar bounds on higher derivatives as for the first derivatives. We will only provide a heuristic proof here, for the complete proof please see [15, Section 2.6].

Theorem 2.17. *If κ is bounded by K on the time interval on the time interval $[0, t_0)$ with $0 < t_0 < K^{-2}$, then $\kappa^{(m)}$ is bounded on $(0, t_0)$ by $C_m K t^{-\frac{m}{2}}$ for each $m \geq 1$, where $C_m > 0$ is a constant.*

Proof. If we consider the function

$$\Psi_n := \sum_{i=0}^{n} a_i t^i (\kappa^{(i)})^2,$$

where $(a_n)_n$ is sequence of constants not yet determined.
We can deduce for $n \geq 0$

$$\Psi_n = \Psi_{n-1} + a_n t^n (\kappa^{(n)})^2.$$

Consider $(\Psi_n)_t - (\Psi_n)_{ss}$ and choose suitable a_1 such that (2.10) holds. Then there exist positive constants a_1 and A_1 such that

$$(\Psi_1)_t - (\Psi_1)_{ss} \leq -2a_1 t^1 (\kappa^{(2)})^2 + A_1 K^4.$$

By induction we can show that for each $n \geq 1$ there exist positive a_n and A_n satisfying

$$(\Psi_n)_t - (\Psi_n)_{ss} \leq -2a_1 t^1 (\kappa^{(n+1)})^2 + A_n K^4.$$

By the comparison principle for ODE

$$a_n t^n (\kappa^{(n)})^2 \leq \Psi_n \leq a_o K^2 + A_n K^4 t \leq (a_0 + A_n) K^2.$$

Dividing by a_n and taking the square root

$$t^{\frac{n}{2}} |\kappa^{(n)}| \leq C_n K$$

for $t \in [0, t_0)$, where $C_n := \sqrt{(a_0 + A_n)/a_n}$ is a constant depending on n. $\square$

Theorem 2.18. *Let X be a smooth solution to the curve shortening flow. Then X continues to exist as long as the curvature is finite. Or equivalently,*

$$\limsup_{t\to T} \sup_{M^1\times\{t\}} |\kappa| = \infty,$$

where T is the maximal existence time.

Proof. The main idea is to show that, as long as the curvature is bounded by a finite constant $K > 0$ on $t \in [0, T)$, we will always be able to extend X to a later time. First we choose a suitable parametrisation such that $|X'(x, 0)| = 1$.

Lemma 2.19. *For all $(x, t) \in M^1 \times [0, T)$*

$$e^{-K^2T} \leq |X'(x,t)| \leq 1.$$

Proof. Using Remark 2.3 to compute that

$$\begin{aligned}\partial_t |X'| &= \frac{X'}{|X'|}(X_t)' = T|X'|(X_t)_s = T|X'|(-\kappa N)_s \\ &= -T|X'|(\kappa_s N + \kappa N_s) = -T|X'|(\kappa_s N + \kappa^2 T) \\ &= -\kappa^2 |X'|.\end{aligned}$$

This implies $-K^2 \leq -\kappa^2 \leq \partial_t \log |X'| \leq 0$, integrating yields

$$-K^2T \leq -K^2 t \leq \log|X'(x,t)| \log|X'(x,0)| \leq 0.$$

Taking the exponential gives us the result. □

For the bounds on higher derivatives (with respect to the fixed spatial parameter) we claim:

Lemma 2.20. *For each $k \geq 1$ there exist $A_k > 0$ and $B_k > 0$ such that*

$$|X^{(k)}| \leq A_k,\ |\partial_t X^{(k)}| \leq B_k.$$

Now we require a special expression for $\partial_t |X^{(k)}|$, see [15, Lemma 2.14]. Using this expression, Lemma 2.20 follows immediately from Lemma 2.19 and induction on k. From Theorem 2.17 we know that $||X(\cdot, t_2) - X(\cdot, t_1)||_{C^k} \leq C|t_2 - t_1|$ for any $t_1, t_2 \in [0, t_0]$. By the completeness of $C^k(\mathbb{R}/(n\mathbb{Z}), \mathbb{R}^2)$, $X(\cdot, t_n)$ is Cauchy and hence converges to $X(\cdot, T)$ for any $t_n \to T$. Applying the Arzelà-Ascoli theorem we have

$$||X(\cdot,t) - X(\cdot,T)||_{C^k} \leq ||X(\cdot,t) - X(\cdot,t_n)||_{C^k} + ||X(\cdot,t_n) - X(\cdot,tT)||_{C^k} \leq C(T-t),$$

so $X(\cdot, t)$ converges to $X(\cdot, T)$. Thus we have obtained a smooth immersion as $t \to T$. By Theorem 2.15, there exists a smooth solution $X(t)$ on $[T, T + \delta]$. In this way we extend the solution to $[0, T + \delta]$, and the global existence is proved. □

2.4 Grayson's theorem

After Gage and Hamilton proved in [7] that the curve shortening flow shrinks a convex embedded curve in the plane to a point, Grayson added later that all embedded curves become convex in [8]. Eventually the new fact led to the result that curve shortening flow shrinks embedded planar curves to points smoothly, with round limiting shape. However, in this thesis we will not follow the proof that Grayson gave himself, but the approach described in [15].
First we introduce the avoidance principle, which states that two initially disjoint curves will not cross each other in the evolution under the curve shortening flow. Although we will not use it to prove the Grayson's theorem, but it is a beautiful geometric property and the technique required for its proof is commonly used in geometric analysis, especially in some theorems that will come later.

Theorem 2.21 (Avoidance Principle). *Let $X_1 : M_1^1 \times [0,T) \to \mathbb{R}^2$ and $X_2 : M_2^1 \times [0,T) \to \mathbb{R}^2$ be two solutions according to the curve shortening flow, if these two solutions do not intersect each other at initial time, then they will stay disjoint at each $t \in [0,T)$.*

Proof. We consider the length of the shortest lines segment joining X_1 and X_2. If this length is non-decreasing in time, then it will stay positive at any time. Now we define

$$\begin{aligned} d : M_1 \times M_2 \times [0,T) &\to \mathbb{R} \\ (x,y,t) &\mapsto |X_2(x,t) - X_1(y,t)|. \end{aligned}$$

The length of the shortest lines segment at initial time is

$$d_0 := \inf\{d(x,y,0) | (x,y) \in M_1 \times M_2\}.$$

By the compactness of X_1 and X_2 we get $d_0 > 0$. We prove by contradiction that $d(x,y,t) \geq d_0$ at every $t \in [0,T)$, which is equivalent to $de^{\varepsilon(1+t)} > d_0$ for every $\varepsilon > 0$. Suppose the claim is not true, then there exists some $t_0 \in (0,T)$ such that

$$\inf\{d(x,y,t_0)e^{\varepsilon(1+t_0)} | (x,y) \in M_1 \times M_2\} = d_0$$

by the continuity of $\inf\{de^{\varepsilon(1+t)}\}$ in time. Once again, by the compactness of X_1 and X_2, there also exist $(x_0,y_0) \in M^1 \times M^2$ such that $d(x_0,y_0,t_0) = d_0$. If we consider the derivatives of d at (x_0,y_0,t_0), then we have $\frac{\partial de^{\varepsilon(1+t)}}{\partial t} \leq 0$, $\nabla d = 0$ and $\operatorname{Hess} d \geq 0$. These three (in-)equalities will be the keys to finish the proof. Next let T_1, T_2 be the unit tangent vectors and $N_1,\ N_2$ be the unit normal vectors of the parametrised curves $X_1(\cdot\,,t_0), X_2(\cdot\,,t_0)$. Furthermore let $\kappa_1,\ \kappa_2$ be the corresponding curvatures. $\nabla d = 0$ implies:

$$\begin{aligned} 0 = \frac{\partial d}{\partial x} &= \frac{1}{|X_2 - X_1|}\left\langle X_2 - X_1,\ \frac{d(X_2 - X_1)}{dx}\right\rangle = -\left\langle \frac{X_2 - X_1}{|X_2 - X_1|},\ T_1\right\rangle, \\ 0 = \frac{\partial d}{\partial y} &= \frac{1}{|X_2 - X_1|}\left\langle X_2 - X_1,\ \frac{d(X_2 - X_1)}{dy}\right\rangle = \left\langle \frac{X_2 - X_1}{|X_2 - X_1|},\ T_2\right\rangle. \end{aligned} \tag{2.11}$$

If we define $\omega := \frac{X_2 - X_1}{|X_2 - X_1|}$, then ω is perpendicular to $T_1,\ T_2$. Without loss of generality we may assume that $N_1 = N_2 = \omega$ and $T_1 = T_2$, as we can always change the direction of the parametrisation.
We compute next the derivatives of ω to get the second deriviatives of d.

$$\begin{aligned} \frac{\partial \omega}{\partial x} &= \frac{\frac{\partial}{\partial x}|X_2 - X_1| - \frac{\partial}{\partial x}|X_2 - X_1|(X_2 - X_1)}{|X_2 - X_1|^2} \\ &= \frac{-T_1|X_2 - X_1| - \langle \omega,\ -T_1\rangle (X_2 - X_1)}{|X_2 - X_1|^2} \\ &= \frac{-T_1 + \langle \omega,\ T_1\rangle \omega}{|X_2 - X_1|}, \\ \frac{\partial \omega}{\partial y} &= \frac{\frac{\partial}{\partial y}|X_2 - X_1| - \frac{\partial}{\partial y}|X_2 - X_1|(X_2 - X_1)}{|X_2 - X_1|^2} \\ &= \frac{T_2|X_2 - X_1| - \langle \omega,\ T_2\rangle (X_2 - X_1)}{|X_2 - X_1|^2} \\ &= \frac{T_2 - \langle \omega,\ T_2\rangle \omega}{|X_2 - X_1|}, \end{aligned} \tag{2.12}$$

and the second derivatives of d:

$$\begin{aligned} \frac{\partial^2 d}{\partial x^2} &= -\left\langle \frac{\partial \omega}{\partial x},\ T_1\right\rangle - \left\langle \omega,\ \frac{\partial T_1}{\partial x}\right\rangle = \frac{\langle T_1 - \langle \omega,\ T_1\rangle \omega,\ T_1\rangle}{|X_2 - X_1|} + \langle \omega,\ \kappa_1 N_1\rangle, \\ \frac{\partial^2 d}{\partial y^2} &= \left\langle \frac{\partial \omega}{\partial y},\ T_2\right\rangle + \left\langle \omega,\ \frac{\partial T_2}{\partial y}\right\rangle = \frac{\langle T_2 - \langle \omega,\ T_2\rangle \omega,\ T_2\rangle}{|X_2 - X_1|} + \langle \omega,\ -\kappa_2 N_2\rangle, \\ \frac{\partial^2 d}{\partial x \partial y} &= -\left\langle \frac{\partial \omega}{\partial y},\ T_1\right\rangle - \left\langle \omega,\ \frac{\partial T_1}{\partial y}\right\rangle = -\frac{\langle T_1 - \langle \omega,\ T_1\rangle \omega,\ T_2\rangle}{|X_2 - X_1|}. \end{aligned} \tag{2.13}$$

Combining with our assumptions at (x_0, y_0, t_0): $T_1 = T_2,\ N_1 = N_2 = \omega$ we obtain

$$\frac{\partial^2 d}{\partial x^2} = \frac{1}{d} + \kappa_1, \quad \frac{\partial^2 d}{\partial y^2} = \frac{1}{d} - \kappa_2, \quad \frac{\partial^2 d}{\partial x \partial y} = -\frac{1}{d}. \tag{2.14}$$

Hess $d \geq 0$ gives

$$0 \leq \left(\frac{\partial d}{\partial x} + \frac{\partial d}{\partial y}\right)^2 = \frac{\partial^2 d}{\partial x^2} + \frac{\partial^2 d}{\partial y^2} + 2\frac{\partial^2 d}{\partial x \partial y} = \frac{1}{d} + \kappa_1 + \frac{1}{d} - \kappa_2 - 2\frac{1}{d} = \kappa_1 - \kappa_2. \quad (2.15)$$

Finally we compute the time derivative of $de^{\varepsilon(1+t)}$ at (x_0, y_0, t_0) and use the inequality $\frac{\partial de^{\varepsilon(1+t)}}{\partial t} \leq 0$,

$$\begin{aligned} 0 &\geq \frac{\partial d}{\partial t} e^{\varepsilon(1+t)} + d\frac{\partial e^{\varepsilon(1+t)}}{\partial t} \geq \left\langle \frac{X_2 - X_1}{|X_2 - X_1|},\ \frac{\partial}{\partial t}(X_2 - X_1) \right\rangle + \varepsilon d e^{\varepsilon(1+t)} \\ &> \left\langle \omega,\ \frac{\partial}{\partial t}(T_2 - t_1) \right\rangle = \left\langle \omega,\ \frac{\partial}{\partial t}(-\kappa_2 N_2 + \kappa_1 N_1) \right\rangle = \kappa_1 - \kappa_2. \end{aligned} \quad (2.16)$$

But this is a contradiction to (2.15), which means our assumption is wrong and $de^{\varepsilon(1+t)} > d_0$ for every $\varepsilon > 0$. If we choose ε sufficiently small, then $d \geq d_0 > 0$, and hence X_1 and X_2 will stay disjoint at any $t \in [0, t_0]$. □

Now we might wonder how does an embedded curve behave in the evolution. The result is not surprising: if a curve does not intersect itself at initial time, then it will remain embedded during the whole evolution.

Theorem 2.22 (Embeddedness is preserved)**.** *Let $X : M^1 \times [0, t_0] \to \mathbb{R}^2$ be a solution to the curve shortening flow which is smooth and an embedding at $t = 0$. Then $X(\cdot, t)$ is an embedding for every $t \in [0, t_0]$.*

Proof. Due to the compactness of M^1, it is enough to show that $X(\cdot, t)$ is injective for all $t \in [0, t_0]$. We consider the extrinsic distance defined by

$$\begin{aligned} d : M_1 \times M_2 \times [0, t_0] &\to \mathbb{R} \\ (x, y, t) &\mapsto |X(y, t) - X(x, t)|. \end{aligned}$$

It is obvious to see that d vanishes on the "diagonal" set $\{(x, x) | x \in M^1\}$, so d cannot stay positive. We want to show that d remains positive outside the diagonal set, so we use the boundedness of the curvature to control d on a neighbourhood of the diagonal set and apply the maximum principle. We will need the following lemma:

Lemma 2.23. *If $X : M^1 \to \mathbb{R}^2$ is an immersion with curvature bounded by K, and (x, y) is a pair with the arc length $l(x, y) \leq \pi/K$, then*

$$|X(y) - X(x)| \geq \frac{2}{K} \sin\left(\frac{Kl(x, y))}{2}\right). \quad (2.17)$$

Proof. Choose some points $x,\ y$ along the curve and suppose the length of the curve segment joining them satisfies $l(x,y) \leq \pi/K$, choose arc length parameter s such that $s(x) = -l/2, s(y) = l/2$. We compute that

$$\begin{aligned}
\langle X(y) - X(x),\ T(0)\rangle &= \int_{-l/2}^{l/2} \langle T(s),\ T(0)\rangle\, ds \\
&= \int_{-l/2}^{l/2} \langle (-\sin\theta(s), \cos\theta(s))\,,\ (-\sin\theta(0), \cos\theta(0))\rangle\, ds \\
&= \int_{-l/2}^{l/2} \cos\left(|\theta(s) - \theta(0)|\right) ds \\
&\geq \frac{2}{K}\sin\left(\frac{Kl}{2}\right).
\end{aligned}$$

We used in the last step that

$$\cos\left(|\theta(s) - \theta(0)|\right) \leq |\int_0^s \kappa ds| \leq \int_0^s |\kappa| ds \leq K|s| \leq \frac{Kl}{2} \leq \frac{2}{\pi}$$

for every $s \in [-l/2, l/2]$. $\square$

Due to the compactness of $X(\cdot\,, t)$, for all $0 \leq l \leq \pi/K$ we have

$$|X(y,t) - X(x,t)| \geq \frac{2}{K}\sin(\frac{Kl}{2}).$$

Define a set

$$S := \left\{(x,y,t) \in M^1 \times M^1 \times [0, t_0] : l(x,y,t) \geq \frac{\pi}{K}\right\}$$

and use the maximum principle on the boundary

$$\inf\left\{d(x,y,t) : l(x,y,t) = \frac{\pi}{K}\right\} \geq \frac{2}{K}\sin\left\{\frac{\pi}{2}\right\} = \frac{2}{K} > 0.$$

Since we want to apply the maximum principle, we would require a PDE for d. We claim that

$$\begin{aligned}
\frac{\partial d}{\partial t} = -\frac{1}{d}&\left(\left(\frac{\partial d}{\partial s_y}\right)^2 + \left(\frac{\partial d}{\partial s_x}\right)^2 - 2T_x \cdot T_y \frac{\partial d}{\partial s_y}\frac{\partial d}{\partial s_x}\right) \\
&+ \left(\frac{\partial^2}{\partial s_y^2} + \frac{\partial^2}{\partial s_x^2} + 2T_x \cdot T_y \frac{\partial^2}{\partial s_y \partial s_x}\right) d,
\end{aligned} \tag{2.18}$$

where we use the notations $T_x := T(x,t)$ and $T_y := T(y,t)$.
First we compute the time derivative of d:

$$\frac{\partial d}{\partial t} = \left\langle \frac{X(y,t)-X(x,t)}{|X(y,t)-X(x,t)|},\ \frac{\partial}{\partial t}\left(X(y,t)-X(x,t)\right) \right\rangle = \left\langle \omega,\ -\kappa_y N_y + \kappa_x N_x \right\rangle, \quad (2.19)$$

where $\omega := \frac{X(y,t)-X(x,t)}{|X(y,t)-X(x,t)|}$, $N_x := N(x,t)$ and $N_y := N(y,t)$.
Taking the derivatives with respect to s_y, we have

$$\frac{\partial d}{\partial s_y} = \left\langle \frac{X(y,t)-X(x,t)}{|X(y,t)-X(x,t)|},\ T_y \right\rangle = \left\langle \omega,\ T_y \right\rangle.$$

Next we compute the spatial derivatives of ω:

$$\frac{\partial \omega}{\partial s_x} = \frac{-T_x + \langle \omega,\ T_x\rangle\, \omega}{|X(y,t)-X(x,t)|}, \quad \frac{\partial \omega}{\partial s_y} = \frac{T_y - \langle \omega,\ T_y\rangle\, \omega}{|X(y,t)-X(x,t)|}. \quad (2.20)$$

Then we have

$$\frac{\partial^2 d}{\partial s_y^2} = \left\langle \frac{\partial \omega}{\partial s_y},\ T_y \right\rangle + \left\langle \omega,\ \frac{\partial T_y}{\partial s_y} \right\rangle = \frac{1 - \langle \langle \omega,\ T_y\rangle\, \omega,\ T_y\rangle}{|X(y,t)-X(x,t)|} - \langle \kappa_y N_y,\ T_y\rangle.$$

Similarly we can compute the s_x-derivatives,

$$\frac{\partial d}{\partial s_x} = \langle \omega,\ -T_x\rangle, \qquad \frac{\partial^2 d}{\partial s_x^2} = \frac{1 - \langle \langle \omega,\ T_x\rangle\, \omega,\ T_x\rangle}{|X(y,t)-X(x,t)|} + \langle \kappa_x N_x,\ T_x\rangle.$$

Moreover,

$$\frac{\partial^2 d}{\partial s_y \partial s_x} = -\frac{\langle T_y - \langle \omega,\ T_y\rangle\, \omega,\ T_x\rangle}{|X(y,t)-X(x,t)|}. \quad (2.21)$$

Combining all the above derivatives, (2.18) is proved.
Applying the maximum principle one more time, it follows that

$$d(x,y,t) \geq \min\left\{ \inf\left\{ d(x,y,t) : l(x,y,0) \geq \frac{\pi}{K} \right\}, \frac{2}{K} \right\} > 0.$$

Thus $d(x,y,t) > 0$ for all $(x,y,t) \in M^1 \times M^1 \times [0,t_0]$ with $x \neq y$. $\square$

We found in the above proof that there is a positive lower bound for d outside the diagonal, but this bound depends on an assumed curvature bound. Huisken gave an improvement of this distance bound in [11], and the refined estimates are only in terms of two lengths: the arc length and the total length of the curve.

Theorem 2.24 (Huisken's Distance Comparison Estimate)**.** *Let $X : M^1 \times [0,T] \to \mathbb{R}$ be a smooth embedded solution to the curve shortening flow. Define the function $Z : (M^1 \times M^1 \setminus \{(x,x) | x \in M^1\}) \times [0,T) \to \mathbb{R}$ by*

$$Z(x,y,t) := \frac{L(t)}{d(x,y,t)} \sin\left(\frac{\pi l(x,y,t)}{L(t)}\right), \tag{2.22}$$

where $l(x,y,t)$, $d(x,y,t)$ are defined as in Theorem 2.22 and $L(t)$ is the total length of the curve at time t. Then

$$\frac{\partial\left(\sup Z(x,y,t)\right)}{\partial t} \leq 0,$$

and the equality holds if and only if X is a round circle.

Proof. If X is initially a round circle, then it will keep shrinking to smaller circles. From the relation $d = \frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)$ we know that Z is constant in the process, and hence the first derivatives of Z are zero. Now we assume that X is not a round circle and show that $\sup Z$ is strictly decreasing in time, i.e. for any $t_0 \in [0,T)$ there is no $t_1 \in [t_0, T)$ such that

$$\sup_{x,y \in M^1} Z(x,y,t_1) = \sup_{\substack{x,y \in M^1 \\ t \in [t_0,t_1]}} Z(x,y,t).$$

Suppose there exist such t_0 and t_1. Then there exists a pair $(x_1,y_1) \in M^1 \times M^1$ such that $Z(x_1,y_1,t_1) = \sup\{Z(x,y,t) \ : \ x,y \in M^1, \ t \in [t_0,t_1]\}$. Note that $x_1 \neq y_1$, since otherwise X would be a round circle (see [15, Lemma 3.8, Lemma 3.9].) Evaluating at point (x_1,y_1,t_1) we have $\frac{\partial Z}{\partial t} \geq 0$ and $\operatorname{Hess} Z \leq 0$. Similar to the proof of Theorem 2.21, we use these two inequalities to derive a contraction.

Recall that

$$\frac{\partial L}{\partial_t} = \int_{M^1} -\kappa^2 ds, \quad \frac{\partial l(x,y,t)}{\partial_t} = \int_x^y -\kappa^2 ds.$$

Taking the time derivative, we find that

$$\frac{\partial Z}{\partial t} = \frac{\frac{\partial}{\partial_t}\left(L\sin\left(\frac{\pi l}{L}\right)\right)d - \frac{\partial d}{\partial_t}\left(L\sin\left(\frac{\pi l}{L}\right)\right)}{d^2}.$$

Multiplying both sides by d gives

$$
\begin{aligned}
d\frac{\partial Z}{\partial t} &= \frac{\partial}{\partial t}\left(L\sin\left(\frac{\pi l}{L}\right)\right) - \frac{L}{d}\sin\left(\frac{\pi l}{L}\right)\frac{\partial d}{\partial t} \\
&= \frac{\partial L}{\partial t}\sin\left(\frac{\pi l}{L}\right) + L\cos\left(\frac{\pi l}{L}\right)\frac{\pi l\frac{\partial L}{\partial t} - \pi L\frac{\partial l}{\partial t}}{L^2} - Z\frac{\partial d}{\partial t} \\
&= Z\left\langle \omega,\ \kappa_y N_y - \kappa_x N_x\right\rangle - \int_{M^1}\kappa^2 ds\left(\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right)\right) \\
&\quad - \pi\cos\left(\frac{\pi l}{L}\right)\int_x^y \kappa^2 ds,
\end{aligned}
\tag{2.23}
$$

where $\omega := \frac{X(y,t)-Y(x,t)}{|X(y,t)-Y(x,t)|}$ and the last equality follows from (2.19).

Assume without loss of generality that $l \in [0, L/2]$, we have then $\pi l/L \in [0, \pi/2]$. Therefore $\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right) \geq 0$ and $\cos\left(\frac{\pi l}{L}\right) \geq 0$, hence the second term and the third term are non-positive. We focus on the first term now. First we give an orientation to the curve so that $s(t)$ increases from x to y along the shorter path, which means $\frac{\partial l}{\partial x} = -1$ and $\frac{\partial l}{\partial y} = 1$. Then the first spatial derivatives are

$$
\begin{aligned}
d\frac{\partial Z}{\partial x} &= d\left(\frac{-L\frac{\partial d}{\partial x}}{d^2}\sin\left(\frac{\pi l}{L}\right) + \frac{L}{d}\cos\left(\frac{\pi l}{L}\right)\frac{\pi\frac{\partial l}{\partial x}}{L}\right) \\
&= \frac{-L\frac{\partial d}{\partial x}}{d}\sin\left(\frac{\pi l}{L}\right) - \pi\cos\left(\frac{\pi l}{L}\right) \\
&= -Z\frac{\partial d}{\partial x} + \pi\cos\left(\frac{\pi l}{L}\right) = Z\left\langle \omega,\ T_x\right\rangle - \pi\cos\left(\frac{\pi l}{L}\right), \\
d\frac{\partial Z}{\partial y} &= -Z\left\langle \omega,\ T_y\right\rangle + \pi\cos\left(\frac{\pi l}{L}\right).
\end{aligned}
\tag{2.24}
$$

We compute the second derivatives

$$
\begin{aligned}
d\frac{\partial^2 Z}{\partial x^2} &= Z\left\langle \frac{\partial \omega}{\partial x},\ T_x\right\rangle - Z\left\langle \omega,\ \kappa_x N_x\right\rangle + \pi\sin\left(\frac{\pi l}{L}\right)\frac{\pi}{L}\frac{\partial l}{\partial x} \\
&= -\frac{Z}{d}\left(1 - \left\langle \omega,\ T_x\right\rangle^2\right) - Z\left\langle \omega,\ \kappa_x N_x\right\rangle - \frac{\pi^2}{L}\sin\left(\frac{\pi l}{L}\right), \\
d\frac{\partial^2 Z}{\partial y^2} &= -\frac{Z}{d}\left(1 - \left\langle \omega,\ T_y\right\rangle^2\right) + Z\left\langle \omega,\ \kappa_x N_x\right\rangle - \frac{\pi^2}{L}\sin\left(\frac{\pi l}{L}\right), \\
d\frac{\partial^2 Z}{\partial x \partial y} &= \frac{Z}{d}\left(\left\langle T_x,\ T_y\right\rangle - \left\langle \omega,\ T_x\right\rangle\left\langle \omega,\ T_y\right\rangle\right) + \frac{\pi^2}{L}\sin\left(\frac{\pi l}{L}\right).
\end{aligned}
\tag{2.25}
$$

Since the first derivatives of Z vanish, we have

$$Z\left\langle \omega,\ T_x\right\rangle = \pi\cos\left(\frac{\pi l}{L}\right) = Z\left\langle \omega,\ T_y\right\rangle .$$

There are two possibilities:

1. T_x is parallel to T_y;
2. T_x and T_y are bisected by ω.

Case 1: Suppose the two tangent vectors are parallel. Then the first term in the last line of (2.23) becomes zero and the time derivative is negative, thus

$$d\left(\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2\right) Z < 0.$$

This is a contradiction to the choice of (x_1, y_1, t_1).
Case 2: Suppose the two tangent vectors are bisected by ω. Denote the angle between T_x and ω by θ. Similarly we derive

$$\begin{aligned} d\left(\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2\right) Z = &- \int_{M^1} \kappa^2 ds \left(\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right)\right) \\ &- \pi\cos\left(\frac{\pi l}{L}\right)\int_x^y \kappa^2 ds + \frac{4\pi^2}{L}\sin\left(\frac{\pi l}{L}\right) \\ &- \frac{Z}{d}(2\sin^2\theta + 2\cos 2\theta - 2\cos^2\theta) > 0, \end{aligned} \tag{2.26}$$

by $\int_{M^1}\kappa^2 ds > \frac{1}{L}\left(\int_{M^1}\kappa ds\right)^2 = \frac{4\pi^2}{L}$, $\int_x^y \kappa^2 ds \geq \frac{4\theta^2}{l}$ and $0 \leq \cos\theta \leq \left(\frac{\pi l}{L}\right)$. This also contradicts to the fact that we have chosen a supreme point.

Therefore neither of these two cases is possible, our assumption that $\sup Z$ does not decrease strictly is wrong. □

Compared to Theorem 2.23, Huisken's estimate produces lower bounds for d which do not depend on a curvature bound. However, this estimate is still not strong enough to prove Grayson's theorem. If we want to show that the solution continues to exist whenever the length remains positive, we would want to use Theorem 2.15. Therefore we would need a finite bound for curvature, given that the length is positive. Andrews and Bryan introduced in [13] a short and direct proof of Grayson's theorem, using a refinement of the Huisken's estimate.

Theorem 2.25 (Andrews-Bryan)**.** *Let $X : M^1 \times [0,T) \to \mathbb{R}^2$ be smooth and evolved by the curve shortening flow. If the initial curve satisfies*

$$d(x,y,0) \geq \frac{L(0)}{a} \arctan\left(\frac{a}{\pi}\sin\left(\frac{\pi l(x,y,0)}{L(0)}\right)\right) \tag{2.27}$$

for some positive a and for all $(x,y) \in M^1 \times M^1$, then there is a global lower bound for all $t \in [0,T)$:

$$d(x,y,t) \geq \frac{L(t)e^{4\pi^2\tau(t)}}{a} \arctan\left(\frac{a}{\pi e^{4\pi^2\tau(t)}}\sin\left(\frac{\pi l(x,y,t)}{L(t)}\right)\right), \tag{2.28}$$

where $\tau(t) := \int_0^t \frac{1}{L^2}dt$.

Proof. The idea is to study the chord-arc profile of the curve. By Lemma 2.23 and Taylor expansion we know that a remarkably strong lower bound for the chord-arc profile implies a upper bound for curvature. By using a similar argument we used in proving Theorem 2.24, we get the desired bound for chord-arc profile. See [15, Section 3.4] for the detailed proof. □

Remark 2.26. *In fact, there always exists a constant $a > 0$ for a smooth embedding with initial data X_0 such that the condition (2.27) holds for all x and y.*

We obtain the desired upper bounds for curvature:

Corollary 2.27.

$$\kappa^2(x,t) \leq \left(\frac{2\pi}{L(t)}\right)^2\left(1 + \frac{2a^2}{\pi^2}e^{-8\pi^2\tau(t)}\right) \; \forall x \in M^1, t \in [0,T).$$

Theorem 2.28 (Grayson's Theorem)**.** *Let $X_0 : M^1 \to \mathbb{R}^2$ be a smooth embedding, M^1 is compact and connected. Then the solution X to the curve shortening flow with $X(0) = X_0$ exists on a maximal interval $[0,T)$. In particular, the solution converges to a limiting point p as $t \to T$. Moreover, if we rescale X in space and time by*

$$\widetilde{X}(\cdot\,,t) := \frac{X(\cdot\,,t) - p}{\sqrt{2(T-t)}},$$

then $\widetilde{X}$ converges to a limit $\widetilde{X}_T$ in the C^∞ sense with image S^1 about the origin.

Proof. This proof consists of five parts:

1. The length converges to zero as $t \to T$;
2. $\tau(t)$ approaches infinity as $t \to T$;
3. The curvature converges to a constant and hence the curve is becoming circular;
4. The curve converges to a point $p \in \mathbb{R}^2$;
5. All the derivatives of the rescaled embedding $\widetilde{X}$ are bounded, then $\widetilde{X}$ converges.

Part 1: As $L_t = \int_{M^1} -\kappa^2 ds < 0$, $L(t)$ is decreasing in time. Suppose $L(t)$ does not approach zero, then there is a positive bound $L*$ such that $L(t) \geq L* > 0$ for all $t \in (0, T]$. It follows from Remark 2.26 and Corollary 2.27 that $\kappa^2 \leq \frac{C}{L*^2}$, where C is a positive constant. By Theorem 2.18 the solution will continue to exist after the maximal time, this is a contradiction.

Part 2:

Lemma 2.29. *[15, cf. Lemma 3.21]. For all $t \in [0, T)$:*

$$2\pi\sqrt{2(T-t)} \leq L(t) \leq 2\pi\sqrt{2(1+\frac{2a^2}{\pi^2})(T-t)}.$$

By the definition of τ we obtain

$$-\frac{1}{8\pi^2+16a^2}\log\left(1-\frac{t}{T}\right) \leq \tau(t) \leq -\frac{1}{8\pi^2}\log(1-\frac{t}{T}). \tag{2.29}$$

Thus $\tau(t) \to \infty$ as $t \to T$. *Part 3:*

Lemma 2.30. *[15, cf. Lemma 3.23] If we set $C := \frac{2a^2}{\pi^2}T^{-\frac{1}{1+\frac{2a^2}{\pi^2}}}$, then*

$$2\pi\sqrt{2(T-t)} \leq L(t) \leq 2\pi\sqrt{2(T-t)\left(1+C(T-t)^{\frac{1}{1+\frac{2a^2}{\pi^2}}}\right)}.$$

From this lemma we find finer curvature and curve length bounds.

Corollary 2.31. *[15, cf. Corollary 3.24] There is some finite C such that*

$$\kappa^2 \leq \left(\frac{2\pi}{L}\right)(1+C(T-t)), \quad L \leq 2\pi\sqrt{2(T-t)}\,(1+C(T-t)). \tag{2.30}$$

If we can find bounds on the rescaled embeddings and on all its derivatives, then we will be able to find the limiting curve. For $k \geq 1$ we can compute the k-th spatial derivative of the curvature on rescaled curve: $(T-t)^{(k+1)/2}\frac{\partial^k \kappa}{\partial s^k}$. So we need the following lemma.

Lemma 2.32. *There exists constants A_k for each $k \geq 1$ such that*

$$\left|\frac{\partial^k \kappa}{\partial s^k}\right|^2 \leq \frac{A_k}{(T-t)^{1+k}}.$$

Proof. Using Corollary 2.27 and Lemma 2.29 we find that there is A_0 such that $|\kappa| \leq A_0(T-t)^{-1/2}$. If we consider some time interval with length $\alpha/(1+A_0^2)$ we get $|\kappa| \leq \sqrt{(1+A_0^2)/\alpha}$. Applying Theorem 2.17 we have

$$\left|\frac{\partial^k \kappa}{\partial s^k}\right| \leq A_k \sqrt{(1+A_0^2)/\alpha}(t-T+\alpha)^{-k/2}.$$

Fix a t and choose $\alpha = \min\{T,\ (1+A_0^2)(T-t)/A_0^2\}$, then

$$\left|\frac{\partial^k \kappa}{\partial s^k}\right| \leq A_k \max\left\{\sqrt{(1+A_0^2/T)}t^{-k/2},\ (A_0^2/(T-t))^{(k+1)/2}\right\}.$$

Taking the square on both sides, the lemma is proved. □

We have found bounds for all derivatives of curvature on the rescaled curve, now we are ready to seek a limit for the curvature on $\widetilde{X}_t$.

Lemma 2.33. *There exists positive C such that $\left|\kappa\sqrt{2(T-t)}-1\right| \leq C\sqrt{T-t}$.*

Proof. First consider the integral. Using Corollary 2.31 we have

$$\begin{aligned}\int_{M^1}\left|\kappa\sqrt{2(T-t)}-1\right|^2 ds &= 2(T-t)\int_{M^1}\kappa^2 ds - 2\sqrt{2(T-t)}\int_{M^1}\kappa ds + L \\ &\leq \frac{8\pi^2}{L}\left(T-t+C(T-t)^2\right) - 4\pi\sqrt{2(T-t)} + L \\ &\leq LC(T-t).\end{aligned}$$

It follows from the interpolation inequality that

$$\left|\kappa\sqrt{2(T-t)}-1\right| \leq C(T-t)^{1/2}.$$

□

From this lemma we find that the curvature on $\widetilde{X}$ approaches 1 as $t \to T$. We use the interpolation inequality to deduce a similar bound for higher derivatives of curvature on $\widetilde{X}_t$.

Lemma 2.34. *[15, cf. Lemma 3.27] For each $k \geq 1$ there exists finite constant B_k such that*

$$(T-t)^{\frac{1+k}{2}} \left| \frac{\partial^k \kappa}{\partial s^k} \right| \leq B_k (T-t)^{\frac{1}{4}}.$$

We can deduce now that all the derivatives of the curvature on $\widetilde{X}_t$ decays to zero as $t \to T$.

Part 4: Fix $u_0 \in M^1$ arbitrarily. It follows from Lemma 2.33 that for all $t_1 < t_2 < T$

$$|X(u_0, t_2) - X(u_0, t_1)| \leq \int_{t_1}^{t_2} |\kappa(z_0, t)| dt \leq C\sqrt{T - t_1}.$$

Take $t \to T$, then $X(u_0, t)$ is Cauchy and hence has a limit $p \in \mathbb{R}^2$, with $|X(u_0, t) - p| \leq C\sqrt{T-t}$. By Corollary 2.31 $|X(u,t) - X(u_0,t)| \leq L(t)/2 \leq C\sqrt{T-t}$ for arbitrary $u \in M^1$. Then $|X(u,t) - p| \leq |X(u,t) - X(u_0,t)| + |X(u_0,t) - p| \leq C\sqrt{T-t}$, $X(u,t)$ converges uniformly to p.

Part 5:

Lemma 2.35. *There exists $\widetilde{C} > 0$ such that $|\widetilde{X}'| \geq \widetilde{C}$, and for all $k \geq 1$ there exists C_k such that $|\widetilde{X}^{(k)}| \leq C_k$ and $|\partial_t \widetilde{X}^{(k)}| \leq C_k (T-t)^{-\frac{3}{4}}$.*

Proof. Since $\widetilde{X}' = \frac{X'}{\sqrt{T-t}}$ we have

$$\partial_t \widetilde{X}' = -|\widetilde{X}'| \left(\kappa_s N + \left(\kappa^2 - \frac{1}{2(T-t)} \right) T \right).$$

Then by Lemma 2.33

$$C \leq \partial_t \log |\widetilde{X}'| = -\left(\kappa^2 - \frac{1}{2(T-t)} \right) \leq 0,$$

so $\log |\widetilde{X}'|$ is uniformly bounded. For the higher derivatives, we use the trick in the proof of Theorem 2.18 and prove by induction. ∎

It follows from Lemma 2.35 that $\widetilde{X}'$ converges to a limit $Y(\cdot,T)$ in C^∞ with $|\widetilde{X}_t - Y| \le C(T-t)^{-\frac{1}{4}}$. Define $N := \left(\frac{\widetilde{X}'}{|\widetilde{X}'|}\right)^{\perp}$ and $N_T := \left(\frac{Y}{|Y|}\right)^{\perp}$, then $|N(\cdot,t) - N_T(\cdot)| \le C(T-t)^{\frac{1}{4}}$. So N converges smoothly to N_T. From this we obtain $\left|\partial_t\left(X - \sqrt{2(T-t)N}\right)\right| \le C(T-t)^{-\frac{1}{4}}$, then $|(X - \sqrt{2(T-t)}N) - p| \le C(T-t)^{\frac{3}{4}}$. If we divide both sides by $\sqrt{2(T-t)}$, we proved that $\widetilde{X}$ converges to N_T in the C^∞ sense. Since $N' = \kappa|X'|$ converges to Y, N_T is a smooth diffeomorphism to the 1-sphere about the origin. Therefore we have found the C^∞ limit for $\widetilde{X}$. □

2.5 Convergence to Geodesics

In order to prove the three geodesics theorem, we need to show that if the solution exists for infinite time, then its curvature must converge to zero in the C^∞ norm. This is also part of the main theorem in [9] and implies that the curve approaches a geodesic. We suppose now that the maximal existence time is infinite and prove the result in two steps:

1. Prove that there exists some $L_\infty > 0$ such that $L(t) \to L_\infty$ as $t \to \infty$, where $L(t)$ is the length of the curve at t.
2. Prove that for all $m \ge 0$, the m-th derivative of κ converges to zero in the C^∞ norm as $t \to \infty$.

Since the length L is decreasing in time (see Lemma 2.10), it must have a non-negative limit. Suppose L converges to zero as $t \to T$. As our manifold M^1 is compact, the injectivity radius of M^1 has positive lower bound r. Then the curve is contained in some ball, which is the image set of a ball of radius r under the exponential map. Therefore the length as well as the area enclosed by this curve must be strictly decreasing, and hence the curve shrinks to a point in finite time. This is a contradiction to our assumption that the curve exists for infinite time.

For the second step, we first prove for the case $m = 0$.

Lemma 2.36.

$$\lim_{t\to\infty} \int_{M^1} \kappa^2 ds = 0.$$

Proof. We notice that this integral term is $-L_t$. As L_t will converge at sufficiently large t, it suffices to show there is some bound on the time derivative of $\int_{M^1} \kappa^2 ds$.

We compute that

$$
\begin{aligned}
\partial_t \int_{M^1} \kappa^2 ds &= \int_{M^1} 2\kappa\kappa_t ds + \int_{M^1} \kappa^2 (ds)_t \\
&= \int_{M^1} 2\kappa(\kappa_{ss} + \kappa^3) ds + \int_{M^1} -\kappa^4 ds \\
&= \int_{M^1} \left(2\kappa\kappa_{ss} + \kappa^4\right) ds \\
&= \int_{M^1} \left(-2\kappa_s^2 + \kappa^4\right) ds \\
&\leq \int_{M^1} -2\kappa_s^2 ds + \sup \kappa^2 \int_{M^1} \kappa^2 ds,
\end{aligned} \tag{2.31}
$$

where the second equality follows from Lemma 2.8. Let a, b be the curve lengths such that $\kappa(a,t) = \inf \kappa$ and $\kappa(b,t) = \sup \kappa$. Then we have

$$
\sup|\kappa| - \inf|\kappa| \leq \sup \kappa - \inf \kappa \leq \int_a^b \kappa_s ds \leq \int_{M^1} |\kappa_s| ds. \tag{2.32}
$$

Thus

$$
\begin{aligned}
\sup|\kappa|^2 &\leq \left(\int |\kappa_s| ds + \inf|\kappa|\right)^2 \\
&= (\inf|\kappa|)^2 + \left(\int_{M^1} |\kappa_s| ds\right)^2 + 2\inf|\kappa| \int |\kappa_s| ds.
\end{aligned} \tag{2.33}
$$

Since $\inf|\kappa| \leq \frac{1}{L(t)} \int |\kappa| ds$ and $\left(\int |\kappa_s| ds\right)^2 \leq L(t) \int \kappa_s^2 ds$, if we write $L_0 = L(0)$ and assume without loss of generality $L_0 \geq 1$ and $L_\infty \leq 1$, then we have

$$
\begin{aligned}
\sup \kappa^2 &\leq \left(1 + \frac{1}{L_\infty}\right) \int_{M^1} \kappa^2 ds + (1 + L_0) \int_{M^1} \kappa_s^2 ds \\
&\leq \frac{2}{L_\infty} \int_{M^1} \kappa^2 ds + 2L_0 \int_{M^1} \kappa_s^2 ds.
\end{aligned} \tag{2.34}
$$

By rearranging (2.34) we get a bound on $\int_{M^1} -\kappa_s^2 ds$ and we substitute this term in (2.31)

$$
\begin{aligned}
\partial_t \int_{M^1} \kappa^2 ds &\leq -\frac{1}{L_0} \sup \kappa^2 + \frac{2}{L_0 L_\infty} \int_{M^1} \kappa^2 ds + \sup \kappa^2 \int_{M^1} \kappa^2 ds \\
&= \frac{2}{L_0 L_\infty} \int_{M^1} \kappa^2 ds + \sup \kappa^2 \Big(\int_{M^1} \kappa^2 ds - \frac{1}{L_0}\Big).
\end{aligned} \tag{2.35}
$$

We take t large enough, then $\int_{M^1} \kappa^2 ds = -L_t$ is small enough and hence the growth of it is exponentially bounded by (2.35). Thus it decays to zero exponentially. $\square$

Lemma 2.37.

$$\lim_{t\to\infty} \int_{M^1} \kappa_s^2 ds = 0. \tag{2.36}$$

Proof. Suppose $\int_{M^1} \kappa_s^2 ds$ does not converge to zero. We consider the time derivative of the integral:

$$\begin{aligned} \partial_t \int_{M^1} \kappa_s^2 ds &= \int_{M^1} \kappa_s^2 (ds)_t + \int_{M^1} 2\kappa_s \left(\kappa_t\right)_s ds \\ &= \int_{M^1} -\kappa_s^2 \kappa^2 ds + \int_{M^1} 2\kappa_s \left(\kappa_{ss} + \kappa^3\right)_s ds \\ &= 5 \int_{M^1} \kappa_s^2 \kappa^2 ds - 2 \int_{M^1} \kappa_{ss}^2 ds. \end{aligned} \tag{2.37}$$

The second equality follows from Lemma 2.7(i) and Lemma 2.8, the last equality is obtained by integration by parts. If we can bound the time derivative by a fraction of $\int_{M^1} \kappa_{ss}^2 ds$, then we can conclude that the time derivative also decays to zero exponentially. We assume that $\int_{M^1} \kappa_s^2 ds > C \int_{M^1} \kappa^2 ds$, where C is some constant. Hölder's inequality implies

$$\begin{aligned} \int_{M^1} \kappa_s^2 ds = \int_{M^1} -\kappa_{ss}\kappa ds &\le \left(\int_{M^1} \kappa_{ss}^2 ds\right)^{\frac{1}{2}} \left(\int_{M^1} \kappa^2 ds\right)^{\frac{1}{2}} \\ &< \left(\int_{M^1} \kappa_{ss}^2 ds\right)^{\frac{1}{2}} C^{-\frac{1}{2}} \left(\int_{M^1} \kappa_s^2 ds\right)^{\frac{1}{2}}, \end{aligned} \tag{2.38}$$

which yields

$$\int_{M^1} \kappa_s^2 ds < C^{-1} \int_{M^1} \kappa_{ss}^2 ds. \tag{2.39}$$

Let $\varepsilon > 0$ arbitrarily small, then by Lemma 2.21 there exists a $t_\varepsilon \in [0, T)$ such that $\int_{M^1} \kappa^2 ds < \varepsilon$ for all $t > t_\varepsilon$. We come back to (2.26) and find an estimation for $\int_{M^1} \kappa_s^2 \kappa^2 ds$:

$$\begin{aligned} \int_{M^1} \kappa_s^2 \kappa^2 ds \le \sup(\kappa_s)^2 \int_{M^1} \kappa^2 ds &\le \left(\int_{M^1} |\kappa_{ss}|\right)^2 \varepsilon \\ &\le \varepsilon L_0 \int_{M^1} \kappa_{ss}^2 ds. \end{aligned} \tag{2.40}$$

In the second equality we used $\sup|\kappa_s| \leq \int_{M^1} |\kappa_{ss}|ds$. Hence

$$\partial_t \int_{M^1} \kappa_s^2 ds = (5\varepsilon L_0 - 2) \int_{M^1} \kappa_{ss}^2 ds < -2 \int_{M^1} \kappa_{ss}^2 ds < C' \int_{M^1} \kappa_s^2 ds,$$

where C' is a constant and we have bounded the time derivative of the integral by a fraction of the integral itself. $\square$

Corollary 2.38.

$$\lim_{t\to\infty} \sup \kappa = 0.$$

Proof. It follows from

$$\sup \kappa^2 \leq \left(\int_{M^1} |\kappa_s| ds \right)^2 \leq L_0 \int_{M^1} \kappa_s^2 ds$$

that $\sup \kappa^2$ convergences to zero. Therefore $\sup \kappa$ also decays to zero. $\square$

Until now, we have proved the statement in step 2 for $m = 0$. The convergence of the higher derivatives can be proved in a similar way: first look at the time derivatives of the L^2-norms of $\kappa^{(m)}$, then use integration by parts and Hölder inequality to get

$$\left(\int_{M^1} (\kappa^m)^2 ds \right)^2 \leq \left(\int_{M^1} (\kappa^{m-1})^2 ds \right)^2 \left(\int_{M^1} (\kappa^{m+1})^2 ds \right)^2,$$

and lastly use induction on m.

We conclude that: if a smooth embedded solution to the curve shortening flow survives after finite time, then it will approach an embedded curve with shortest length and curvature zero, which is exactly a geodesic.

Chapter 3

The Three Geodesics Theorem

In Chapter 2 we showed that how the curve shortening flow deforms curves to geodesics, but we still do not know much about the limiting geodesics. In this chapter we will find geodesics on the Riemannian manifold (S^2, g) specifically and prove the three geodesics theorem, using a topological argument.

We consider the space of all simple closed curves Σ, which is also called the loop space, and give it a topology structure to make it a topological space. Since point curves can be seen as finished deformations by the curve shortening flow in finite time, the other curves in Σ will converge to geodesics. Thus it suffices to consider the pair (Σ, Σ_0), where Σ_0 is the space of all point curves. If we prove that this pair can be deformation retracted onto some space which we are more familiar with, we can find the homology classes of (Σ, Σ_0). Let the curve shortening flow apply on a cycle representing a non-trivial homology class, then the cycle converges to a critical point, i.e. the point with the shortest length. The critical point is then a geodesic on (S^2, g).

3.1 The Loop Space

Definition 3.1. *Define* $\Sigma := \{\gamma : S^1 \to S^2 \mid \gamma \textit{ is smooth and simple closed}\}$*,* $\Sigma_0 := \{\gamma : S^1 \to \{pt\} \in S^2\}$ *and* $\Sigma_l := \{\gamma \in \Sigma \mid L(\gamma) \leq l\}$ *for* $l \geq 0$.

Obviously, all point curves are also smooth and simple closed. Thus $\Sigma^0 \subset \Sigma$. Let g be a smooth Riemannian metric and ρ be the metric induced by g. Now, we define a topology in Σ by:

$$d(\gamma_1,\gamma_2) := \sup_{x\in\gamma_1}\min_{y\in\gamma_2}\rho(x,y) + \sup_{y\in\gamma_2}\min_{x\in\gamma_1}\rho(y,x) + |L(\gamma_1) - L(\gamma_2)|,^{1}$$

where L denotes the length functional.
This is a topological metric: Let $\gamma_1,\ \gamma_2,\ \gamma_3 \in \Sigma$,

1. $d(\gamma_1,\gamma_1) = 0$;
2. (Positivity) $d(\gamma_1,\gamma_2) > 0$, for $\gamma_1 \neq \gamma_2$;
3. (Symmetry) $d(\gamma_1,\gamma_2) = d(\gamma_2,\gamma_1)$;
4. (Triangle inequality) Since $|L(\gamma_1) - L(\gamma_3)| \leq |L(\gamma_1) - L(\gamma_2)| + |L(\gamma_2) - L(\gamma_3)|$ it is left to show that $d'(\gamma_1,\gamma_2) := \sup_{x\in\gamma_1}\min_{y\in\gamma_2}\rho(x,y) + \sup_{y\in\gamma_2}\min_{x\in\gamma_1}\rho(y,x)$ satisfies the triangle inequality: Let $x \in \gamma_1,\ y \in \gamma_2$ arbitrary. Choose $z_1,\ z_2 \in \gamma_3$ such that $d'(x,z_1) = \min_{z\in\gamma_3} d'(x,z)$ and $d'(y,z_2) = \min_{z\in\gamma_3} d'(y,z)$. It follows that

$$\begin{aligned}
d'(\gamma_1,\gamma_3) &= \sup_{x\in\gamma_1}\min_{z\in\gamma_3}\rho(x,z) + \sup_{z\in\gamma_3}\min_{x\in\gamma_1}\rho(z,x)\\
&\leq \sup_{x\in\gamma_1}\min_{z\in\gamma_3}\left(\rho(x,z_1) + \rho(z_1,y)\right) + \sup_{y\in\gamma_2}\min_{z\in\gamma_3}\left(\rho(y,z_2) + \rho(z_2,x)\right)\\
&\leq \sup_{x\in\gamma_1}\min_{z\in\gamma_3}\rho(x,z_1) + \sup_{x\in\gamma_1}\min_{z\in\gamma_3}\rho(z_1,y) + \sup_{y\in\gamma_2}\min_{z\in\gamma_3}\rho(y,z_2) + \sup_{y\in\gamma_2}\min_{z\in\gamma_3}\rho(z_2,x)\\
&\leq \sup_{x\in\gamma_1}\min_{z\in\gamma_3}\rho(x,z) + \sup_{x\in\gamma_1}\min_{z\in\gamma_3}\rho(z,y) + \sup_{y\in\gamma_2}\min_{z\in\gamma_3}\rho(y,z) + \sup_{y\in\gamma_2}\min_{z\in\gamma_3}\rho(z,x)\\
&= d'(\gamma_1,\gamma_2) + d'(\gamma_2 + \gamma_3).
\end{aligned}$$

With this topological metric, the length functional $L : \Sigma \to \mathbb{R}^{\geq 0}$ is continuous and the curve shortening flow evolves continuously.

3.2 Deformation Retraction of (Σ, Σ_0)

We want to deal with something we are familiar with, so it's natural to think about choosing the round metric g_0 as our Riemannian metric and considering great circles on S^2. This is possible, as we are looking for the relative homology and the topological properties will not be effected by the choice of metric. The great circle has the length

[1]This metric is inspired by [10].

2π, so we choose some $\varepsilon > 0$, and prove that (Σ, Σ_0) can be deformation retracted onto $(\Sigma_{2\pi+\varepsilon}, \Sigma_{2\pi-\varepsilon})$.
Naturally we want to express the curve shortening flow as a continuous flow map. Given a curve $\gamma \in \Sigma$, we apply the curve shortening flow on it. By Grayson's theorem, γ deforms continuously. We denote the evolved curve after time t by γ^t. If the flow exists infinitely, then $\phi(\gamma, t) := \gamma^t$ is well-defined on $[0, \infty)$. If γ shrinks to a round point at a finite time, we can always extend $\phi(\gamma, t) := \gamma^t$ to $[0, \infty)$. From this we obtain an expression for curve shortening flow: $\phi_t : \Sigma \to \Sigma$, $\phi_t(x) := \phi(x, t)$, and this map satisfies the properties of a flow.

Lemma 3.2. *$\Sigma_{2\pi+\varepsilon}$ is a strong deformation retract of Σ and Σ_0 is a strong deformation retract of $\Sigma_{2\pi-\varepsilon}$.*

Proof. Let ϕ_t denote the curve shortening flow. Let $T(\gamma)$ be the time when $L(\phi_t(\gamma)) = 2\pi + \varepsilon$. By the properties of curve shortening flow, such $T(\gamma)$ is well defined and, in particular, is unique. Define a function $f : \Sigma \to \Sigma_{2\pi+\varepsilon}$ by

$$f(\gamma) := \begin{cases} \gamma & \text{if } \gamma \in \Sigma_{2\pi+\varepsilon}, \\ \phi_{T(\gamma)}(\gamma) & \text{else.} \end{cases}$$

The continuity of ϕ_t guarantees the continuity of f, and f is a retraction. A deformation retraction is a homotopy between a retraction and id_Σ, now define the homotopy $F : \Sigma \times [0, 1] \to \Sigma$ by

$$F(\gamma, t) := \begin{cases} \gamma & \text{if } \gamma \in \Sigma_{2\pi+\varepsilon}, \\ \phi_{tT(\gamma)}(\gamma) & \text{else.} \end{cases}$$

F satisfies for all $\gamma \in \Sigma : F(\gamma, 0) = \gamma$, $F(\gamma, 1) = f(\gamma) \in \Sigma_{2\pi+\varepsilon}$. And additionally for $\gamma \in \Sigma_{2\pi+\varepsilon}$ it holds $F(\gamma, t) = \gamma \in \Sigma_{2\pi+\varepsilon}$. Thus $\Sigma_{2\pi+\varepsilon}$ is a strong deformation retract of Σ.

The second part can be proved in a similar way. Note that a curve in $\Sigma_{2\pi-\varepsilon}$ will be shrinked to a point under the curve shortening flow, so we denote T^γ as the time when γ approaches a point.
Define similarly $\tilde{f} : \Sigma_{2\pi-\varepsilon} \to \Sigma_0$ by

$$\tilde{f}(\gamma) := \begin{cases} \gamma & \text{if } \gamma \in \Sigma_0, \\ \phi_{T^\gamma}(\gamma) & \text{else,} \end{cases}$$

then $\tilde{f}$ is also continuous. Define the homotopy $\widetilde{F} : \Sigma_{2\pi-\varepsilon} \times [0, 1] \to \Sigma_{2\pi-\varepsilon}$ by

$$\widetilde{F}(\gamma, t) := \begin{cases} \gamma & \text{if } \gamma \in \Sigma_0, \\ \phi_{tT^\gamma}(\gamma) & \text{else.} \end{cases}$$

We can check similarly that $\widetilde{F}$ also satisfies the requirements of a strong deformation retract. $\square$

We have found homotopy equivalences between Σ_0 and $\Sigma_{2\pi-\varepsilon}$ as well as between $\Sigma_{2\pi+\varepsilon}$ and Σ. From now on, we use $H_*(\cdot\,)$ to denote a homology group, in which $*$ is used to represent any non-negative integer. Then we have

$$H_*(\Sigma, \Sigma_0) \cong H_*(\Sigma_{2\pi+\varepsilon}, \Sigma_{2\pi-\varepsilon}). \tag{3.1}$$

3.3 Homology Classes of (Σ, Σ_0)

Due to the result (3.1), we can instead determine the homology classes of $(\Sigma_{2\pi+\varepsilon}, \Sigma_{2\pi-\varepsilon})$. Recall that at the beginning of Section 3.2 we decided to use the round metric g_0 as our Riemannian metric. Additionally, for the rest of this thesis, the (co-)homology groups will always be dealt with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Let $\Lambda \subset \Sigma$ be the set of all circles on S^2, a circle is obtained as a non-empty intersection of S^2 and a plane in $\mathbb{R}^3$. Moreover let $\Delta \subset \Lambda$ be the set of great circles, by a great circle we mean a intersection of S^2 and a plane through the origin. In [5, Chapter 2] Klingenberg introduced how to retract the space of curves of certain length a little larger than a critical value, we absorb his results and conclude that $\Sigma_{2\pi+\varepsilon}$ can be retracted onto Λ. It follows that

$$H_*((\Sigma_{2\pi+\varepsilon}, \Sigma_{2\pi-\varepsilon})) \cong H_*(\Lambda, \Sigma_{2\pi-\varepsilon}). \tag{3.2}$$

Note that Δ is the boundary of Λ modulo $\Sigma_{2\pi-\varepsilon}$. Using the Hesse normal form, one circle $\delta := S^2 \cap P_\delta$ can be uniquely determined by identifying a pair $(N, d) \in S^2 \times [0, 1]$, where P_δ is the plane containing δ, N is the normal unit vector and d is the distance between P_δ and the origin. In the case of a great circle, $d = 0$. It is obvious to see, the great circle determined by $(-N, 0)$ is also δ. Thus we obtain an equivalence relation $(N, 0) \sim (-N, 0)$. Therefore Δ is homeomorphic to $S^2 \times [0, 1]/\,((N, 0) \sim (-N, 0)) = \mathbb{R}P^2$ and by the Thom isomorphism

$$H_*((\Sigma_{2\pi+\varepsilon}, \Sigma_{2\pi-\varepsilon})) \cong H_*(\Lambda, \Sigma_{2\pi-\varepsilon}) \cong H^*(\mathbb{R}P^2).[2] \tag{3.3}$$

We know that $\mathbb{R}P^2$ has three cohomology classes, each of dimension 1, 2, 3. [12, cf. Example 3.12]. In the construction of the Thom isomorphism [3, Section 4] another property of the cohomology classes is implied, which is the subordination.

[2] We denote the cohomology classes, i.e. the dual of homology classes, by $H^*(\cdot\,)$.

Definition 3.3. *Let h_i, h_j be two non-zero homology classes of a closed manifold. If there is a cohomology class ω of positive degree such that $h_i = \omega \cap h_j$, then h_i is said to be subordinate to h_j, where $\cap$ is the cap product.*

Combining (3.1) and (3.3) we conclude that $H_*(\Sigma, \Sigma_0)$ has three subordinate homology classes h_1, h_2 and h_3, respectively of dimension one, two and three.

3.4 Proof of Three Geodesics Theorem

Let $h \in H_*(\Sigma, \Sigma_0))$ be non-zero. Define $\kappa(h) := \inf_{c \in h} \max_{\gamma \in c} L(\gamma)$, where c is a cycle representing h and γ is a curve in c. Furthermore, define the set of critical points K as the curves in Σ whose lengths are invariant under the curve shortening flow.

Lemma 3.4. *Let $h \in H_*(\Sigma, \Sigma_0)$ be non-zero. Define $S := \{\gamma$ is geodesic $\mid L(\gamma) = \kappa(h)\}$ and let U be a neighbourhood of S. Then there exist $\varepsilon > 0$ and a cycle $c \in h$ such that every $\gamma \in c$ satisfies either $\gamma \in U$ or $L(\gamma) < \kappa(h) - \varepsilon$.*

Proof. Choose a cycle $c \in h$ arbitrarily and let all the curves in c evolve under the curve shortening flow. As h is non-zero, the curves in it will never converge to points in finite time. By Section 2.5 the curves will converge to geodesics in the C^∞-sense. Take $\gamma \notin U$, then the curve shortening flow shortens γ on a certain time interval and hence the length of curve get shortened by a certain amount. $\square$

Lemma 3.5. *Let h_i, h_j be two subordinate homology classes and $\omega \cap h_i = h_j$. Then $\kappa(h_i) \leq \kappa(h_j)$ and the equality implies $\omega|_U$ is non-zero, where U is an arbitrary neighbourhood of the critical set at $\kappa(h_i) = \kappa(h_j)$.*

Proof. Let $c \in h_j$, then c also lies in h_i. Then by the definition of the critical level $\kappa(h_i) \leq \kappa(h_j)$. For the equality part we suppose $\omega|_U$ is zero and prove by contradiction. Set $\kappa := \kappa(h_i) = \kappa(h_j)$. If we consider the inclusions $i : U \hookrightarrow \Sigma$, $j : \Sigma \hookrightarrow (\Sigma, U)$ and the induced long exact sequence in cohomology. The exactness implies $\omega = j^*(\eta)$ for some η in $H^*(\Sigma, U)$ if $i^*(\omega) = 0$. Then there is a cocycle $\varphi \in \omega$ such that $\varphi(\Delta)$ vanishes for every $\Delta \in U$. By Lemma 3.4 there exists a cycle $c \in h_n$ and $\varepsilon > 0$ such that either $c \in \Sigma^{\kappa-\varepsilon-\delta}$ for some $\delta > 0$ or $c \in U$. Thus $\varphi \cap c \in h_i$ which contradicts to $\kappa = \kappa(h_i)$. $\square$

Note that Σ is locally contractible (see [5, Section 2.2]), combined with the fact that on S^2 one embedded curve cannot cover another one, we get the following corollary:

Corollary 3.6. *If h_i is subordinate to h_j with $\kappa(h_i) = \kappa(h_j)$, then there exist infinitely many geometrically distinct simple closed geodesics of the same length.*

Proof. The claim follows if we consider a sufficiently small neighbourhood of a curve in K, see [6, (1.4)]. □

Corollary 3.6 leads to three possibilities:

1. If $\kappa(h_1) = \kappa(h_2) = \kappa(h_3)$, then there are infinitely many simple closed geodesics, all of the same length.

2. If $\kappa(h_i) \neq \kappa(h_j)$ for $i, j = 1, 2, 3$, then there exist three different lengths of simple closed geodesics, thus there exist at least three geodesics.

3. If $\kappa(h_i) = \kappa(h_{i+1})$ for $i = 1, 2$, then there exist infinitely many simple closed geodesics of a length L_1 and also geodesics of length L_2.

Therefore we can conclude that there exist at least three geodesics on (S^2, g), the main theorem is proved.

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