Ricci Flow

Let (M^n, g_0) be given.

Defn A Ricci flow on (M^n, g_0) is a family of metrics $(g(t))_{t \in [0, \epsilon)}$ s.t.

$$\partial_t g(t) = -2\mathrm{Ric}(g(t))$$
$$g(0) = g_0 .$$

RF

ϵ depends on M^n and g_0.

Examples :-

① If g_0 is Ricci-flat, i.e, $\mathrm{Ric} = 0 \implies g(t) = g_0$ $\forall t$ is a sol^n. Note that $t \in (-\infty, \infty)$ in this case. e.g. when $M = \mathbb{R}^n$ or flat torus.

② Let g_0 be an Einstein metric, i.e,

$$\mathrm{Ric}(g_0) = \lambda g_0 \text{ for some } \lambda \in \mathbb{R}.$$

Then $g(t) = (1-2\lambda t)\, g_0$ is a sol^n to RF

as $\partial_t g(t) = -2\lambda g_0 = -2\,\mathrm{Ric}(g_0)$

$$= -2\,\mathrm{Ric}\big((1-2\lambda t) g_0\big)$$

$$= -2\,\mathrm{Ric}(g(t)).$$

$\therefore$ $g(t) = 0$ at $t = \dfrac{1}{2\lambda}$.

If $\lambda > 0$ then the solutions are shrinking as $g(t)$ is shrinking from g_0.

$\lambda = 0$ static sol^n

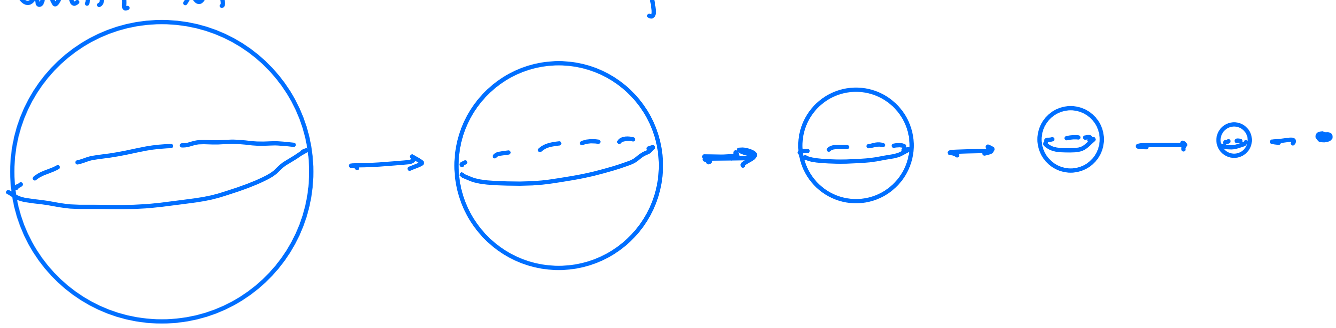
$\lambda < 0$ expanding sol^n.

A concrete case is that of (S^n, g_0). Here

$$\mathrm{Ric}(g_0) = (n-1) g_0 \implies g(t) = (1-2(n-1)t) g_0$$

is a sol^n to RF. and the sol^n exists till $T = \dfrac{1}{2(n-1)}$.

Pictorially, the flow runs by shrinking the sphere until it becomes a point.



Def^n:- • $g(t)$ is called an eternal sol^n if $t \in (-\infty, \infty)$.

• $g(t)$ is an ancient sol^n if $t \in (-\infty, c)$, $c < \infty$.
(e.g. round sphere)

• $g(t)$ is an immortal sol^n if $t \in (\alpha, \infty)$, $\alpha > -\infty$.

Symmetries of RF

* $(M, g(t))_{t \in I}$ is RF $\longrightarrow$ $(M, g_{t-t_0})_{t \in I + t_0}$ is a RF.

* Parabolically rescaling a RF gives another RF
"time scales like $(\text{distance})^2$"
i.e. if $g(t)$ is a RF then

$$\hat{g}(x,t) = \lambda g\left(x, \frac{t}{\lambda}\right), \quad t \in [0, \lambda T] \text{ is also a RF.}$$

as $\partial_t \hat{g}(x,t) = \lambda \cdot \frac{1}{\lambda} \partial_t g\left(x, \frac{t}{\lambda}\right) = -2\,\mathrm{Ric}(g(x, t/\lambda))$
$= -2\,\mathrm{Ric}(\hat{g})$.

* Diffeomorphism invariance. If $\varphi: M \to M$ is a diffeo. and $g(t)$ is a RF then so is $\varphi^* g(t)$.

Ricci flow regarded as a heat eq^n

Def^n:- Local coordinates (x^i) are called <u>harmonic</u> if $\Delta x^i = 0$.

$$\therefore\ 0 = \Delta x^i = g^{jk}(\partial_j \partial_k - \Gamma^l_{jk}\partial_l)x^i = -g^{jk}\Gamma^i_{jk}.$$

Lemma:- For $p \in M$ $\exists$ harmonic coordinates in some nbd. of p.

Lemma:- In harmonic coordinates,

$$-2R_{ij} = \underbrace{\Delta(g_{ij})}_{\text{Laplacian of components}} + \underbrace{Q_{ij}(g^{-1}, \partial g)}_{\text{a term involving quadratic expressions in metric inverse } g^{-1} \text{ and } \partial g.}$$

* **Notation:-** For tensors A, B

$A * B$ would mean "some linear combination of traces of $A \otimes B$ w/ coefficients that do not depend on A or B".

So various contractions b/w tensors whose

precise forms are not important. e.g. $18 A^{kil}_{ij} B^{j}_{lqr}$
$- n!\, A^{lrs}_{qi} B^{k}_{isr}$

$\therefore$ in harmonic coordinates, indeed
$$\partial_t g_{ij} = \Delta(g_{ij}) + Q_{ij}(g^{-1}, \partial g).$$

Proof of the lemma :-

Recall the formula for R_{jk} in local coordinates give

$$-2R_{jk} = -2\left(\partial_q \Gamma^q_{jk} - \partial_j \Gamma^q_{qk} + \Gamma^p_{jk}\Gamma^q_{qp} - \Gamma^p_{qk}\Gamma^q_{jp}\right)$$

$$= -2\Big(\partial_q\Big(\tfrac{1}{2} g^{qr}\big(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}\big)\Big)$$
$$- \partial_j\Big(\tfrac{1}{2} g^{qr}\big(\partial_q g_{kr} + \partial_k g_{qr} - \partial_r g_{kq}\big)\Big)$$
$$+ \Gamma^p_{jk}\Gamma^q_{qp} - \Gamma^p_{qk}\Gamma^q_{jp}\Big)$$

$$= -\partial_q\Big(g^{qr}\big(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}\big)\Big)$$
$$+ \partial_j\Big(g^{qr}\big(\partial_q g_{kr} + \partial_k g_{qr} - \partial_r g_{kq}\big)\Big)$$
$$+ g^{-1} * g^{-1} * \partial g * \partial g$$

(we used the coordinate expression for Γ^k_{ij})

$$= g^{qr}\left(-\underline{\partial_q\partial_j g_{rk}} - \partial_q\partial_k g_{rj} + \underline{\partial_q\partial_r g_{jk}}\right)$$

$$+ g^{qr}\left(\underline{\partial_j\partial_q g_{kr}} + \partial_j\partial_k g_{qr} - \partial_j\partial_r g_{kq}\right)$$

$$+ \quad g^{-1} * g^{-1} * \partial g * \partial g \longrightarrow (*)$$

— terms cancel

— is the $\Delta(g_{jk})$ term on contracting q and r in harmonic coordinates as $\Delta(g_{jk}) = g^{mn}\left(\frac{\partial^2}{\partial x^m \partial x^n} g_{jk} - \Gamma^i_{mn}\frac{\partial g_{jk}}{\partial x^i}\right)$.

The remaining 3 terms can be written in terms of partial derivatives of $\Gamma * g$. e.g.

$$g^{qr}\left(-\partial_q\partial_k g_{rj} + \frac{1}{2}\partial_j\partial_k g_{qr}\right)$$

$$= -g^{qr}\partial_k\left(\Gamma^s_{qr} g_{sj}\right)$$

as

$$-g^{qr}\partial_k\left(\Gamma^s_{qr} g_{sj}\right)$$

$$= -g^{qr}\partial_k\left(\frac{1}{2} g^{sl}\left(\partial_q g_{rl} + \partial_r g_{ql} - \partial_l g_{qr}\right) g_{sj}\right)$$

$$= -g^{qr}\partial_k\left(\frac{1}{2}\left(\partial_q g_{rj} + \partial_r g_{qj} - \partial_j g_{qr}\right)\right)$$

$$= -g^{qr}\left(\frac{1}{2}\left(\partial_k\partial_q g_{rj} + \partial_k\partial_r g_{qj} - \partial_k\partial_j g_{qr}\right)\right)$$

$$= -\partial_k\partial_q g_{rj} + \frac{1}{2}\partial_k\partial_j g_{qr}$$

and the remaining term $g^{qr}\left(-\partial_j\partial_r g_{kq} + \frac{1}{2}\partial_k\partial_j g_{qr}\right)$ in $*$ $= -g^{qr}\partial_j\left(\Gamma^s_{qr}\, g_{sk}\right)$

So $*$ becomes

$$-2R_{jk} = \Delta(g_{jk}) \underline{- g^{qr}\partial_k\left(\Gamma^s_{qr}\, g_{sj}\right) - g^{qr}\partial_j\left(\Gamma^s_{qr}\, g_{sk}\right)}$$
$$+ g^{-1} * g^{-1} * \partial g * \partial g$$

The underlined ———— terms are zero in harmonic coordinates and so

$$-2R_{jk} = \Delta(g_{jk}) + \underbrace{g^{-1} * g^{-1} * \partial g * \partial g}_{Q_{ij}(g^{-1}, \partial g)}$$

$\boxed{/\!/}$